PHOTONICS Research

Limitations of teleporting a qubit via a two-mode squeezed state

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Received 5 December 2018; revised 22 February 2019; accepted 2 March 2019; posted 5 March 2019 (Doc. ID 353382); published 15 April 2019

Recently, a teleportation scheme using a two-mode squeezed state to teleport a photonic qubit, so called a "hybrid" approach, has been suggested and experimentally demonstrated as a candidate to overcome the limitations of all-optical quantum information processing. We find, however, that there exists the upper bound of fidelity when teleporting a photonic qubit via a two-mode squeezed channel under a lossy environment. The increase of photon loss decreases this bound, and teleportation better than this limit is impossible even when the squeezing degree of the teleportation channel becomes infinity. Our result indicates that the hybrid scheme can be valid for fault-tolerant quantum computing only when the photon loss rate can be suppressed under a certain limit.

https://doi.org/10.1364/PRJ.7.0000A7

1. INTRODUCTION

Quantum teleportation is a protocol utilizing an entangled state to transmit an unknown quantum state from one place to another with the aid of a joint measurement and classical communication [1]. It has enabled scientists to explore various applications for quantum information processing since it was first proposed [1]. For example, quantum computation schemes based on linear optics and photodetectors [2] are based on the gate-teleportation protocol [3].

In the original quantum teleportation scheme [1] and its implementation [4], two-qubit entanglement was used to teleport an unknown qubit. One of its crucial elements is the Bell-state measurement that discriminates between four entangled states called "Bell states." In general, its implementation is highly demanding so that only two of the Bell states can be identified using linear optics and photodetectors [5]. As a result, the linear optic quantum computation scheme [2] suffers heavy resource requirements and demanding error thresholds [6,7]. There have been efforts to overcome this obstacle [8–11], but other resource requirements such as added photons [8–10] or squeezing operations [11] are unavoidable.

Another type of quantum teleportation was proposed using continuous-variable (CV) states [12,13] that is also useful for quantum computation [14,15]. It was implemented using a two-mode squeezed state as a nonlocal channel to teleport a coherent state of an unknown amplitude [16]. In this protocol, all measurement results are used, and teleportation with a unit success probability is possible. However, its fidelity must be limited because infinite squeezing, i.e., infinite energy, is required to achieve unit fidelity.

The idea of using continuous variable states for teleporting photonic qubits was suggested in Refs. [17,18], and this "hybrid" approach was experimentally demonstrated for the purpose of efficient quantum information processing [19]. Here, the basic idea is to use the scheme for continuous-variable teleportation [12] to teleport a "discrete" qubit for quantum information processing. In this way, a qubit may be teleported with 100% success probability with some loss of fidelity. Remarkably, it was shown that fault-tolerant quantum computing can be implemented based on this protocol with two-mode squeezed states with 20.5 dB or larger squeezing [20]. In Ref. [20], the authors considered only finite squeezing but did not consider photon loss or detection inefficiency that may affect this protocol. They commented that "they are not expected to change the fundamental result, which is the existence of some finite threshold." However, the loss effect may be more serious than expected because a two-mode squeezed state becomes highly sensitive to photon loss when its squeezing becomes large [21]. This is related to the fact that a two-mode squeezed state becomes a macroscopic superposition that is intrinsically fragile when its squeezing is large [22]. It is thus not immediately clear whether and how much the large initial squeezing can compensate for the photon loss that occurs during the teleportation process to obtain a high fidelity.

In this paper, we find the upper bound of the fidelity when teleporting a photonic qubit via a two-mode squeezed channel under a lossy environment. We employ an alternative formalism expressed in terms of the discrete Fock basis to analyze the hybrid teleportation under a lossy environment, which is different from the formalism introduced in Ref. [23] built upon a pseudoprobability function such as the Wigner function. We first derive a lossy teleportation operator applied to the input state, which gives the output state of the teleportation process via the two-mode squeezed state under photon loss, in analogy to the one for the lossless case [24]. We then show the polarization independence of the fidelity for single-photon qubit teleportation notwithstanding holds from the properties of the derived operator, and finally derive the closed form of the transmission fidelity of the single-photon qubit teleportation that can be applied to any input state. Our result indicates that photon loss in the channel can be compensated for by increasing the squeezing only to a limited extent, but there is a fundamental fidelity limit that is attained with infinite initial entanglement.

2. QUANTUM TELEPORTATION WITH A TWO-MODE SQUEEZED STATE

The initial idea of quantum teleportation with a continuous-variable channel was proposed by Vaidman [13], where the original Einstein–Podolsky–Rosen state [25] was employed as the quantum channel. Braunstein and Kimble suggested a feasible version of continuous-variable teleportation using a two-mode squeezed state with finite squeezing [14] followed by its first experimental demonstration by Furusawa *et al.* [16]. Unlike the original teleportation protocol in the qubit space [1,4], the continuous-variable scheme using a two-mode squeezed state enables one to teleport an arbitrary state including qubits and continuous-variable states.

Before introducing a generalized formulation, we briefly review the teleportation operator formulation introduced in Ref. [26]. In quantum teleportation, input system A and reference system R undergo a projective measurement onto the set of quantum states that can be written in the photon number basis

$$|\beta\rangle_{A,R} = \frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\hat{D}_A(\beta)|n,n\rangle_{A,R}, \tag{1}$$

where $\hat{D}(\beta) \coloneqq \exp(\beta\hat{a}^\dagger - \beta^*\hat{a})$ is the displacement operator with complex number β. The reference system R is entangled with output system B in the two-mode squeezed state, which may be written as

$$|q\rangle_{R,B} = \sqrt{1-q^2}\sum_{n=0}^{\infty}q^n|n,n\rangle_{R,B}. \tag{2}$$

In this setting, provided input state $|\psi\rangle_A$ of system A, the state of system B after the measurement with an outcome corresponding to β can be expressed as

$$|\psi\rangle_B = \sqrt{\frac{1-q^2}{\pi}}\sum_{n=0}^{\infty}q^n|n\rangle_B\langle n|_A\hat{D}_A(-\beta)|\psi\rangle_A. \tag{3}$$

Now, in the scheme analyzed in Ref. [26], upon receiving the measurement outcome β, the receiver displaces the state by $g\beta$, where the g factor is called the gain, which can be chosen freely or can be optimized to maximize the transmission fidelity. The overall teleportation scheme can be expressed compactly as [26]

$$|\psi_{\text{out}}(\beta)\rangle_B = \hat{T}^g_q(\beta)|\psi\rangle_A, \tag{4}$$

where

$$\hat{T}^g_q(\beta) = \sqrt{\frac{1-q^2}{\pi}}\sum_{n=0}^{\infty}q^n\hat{D}_B(g\beta)|n\rangle_B\langle n|_A\hat{D}_A(-\beta). \tag{5}$$

3. MAIN RESULT

A. Lossy Teleportation Operator

The transfer operator defined in the previous section, however, works only in the case of a lossless channel. In a practical situation, the channel system should travel for a long distance between Alice and Bob, and thus unavoidably suffers from photon loss. For the sake of simplicity, we assume symmetric loss for both ends of the two-mode squeezed state. It should be noted that this is not a generic case in a realistic setting. However, the analysis of the symmetric loss yields a useful benchmark in an analytic form that can be helpful for the analysis of an arbitrary loss model. This kind of photon loss can be modelled as a quantum map applied to the two-mode squeezed state $|q\rangle_{AB}$ as

$$\mathrm{Tr}_{E_1E_2}\left(U^{\text{BS}}_{AE_1}U^{\text{BS}}_{BE_2}|q\rangle\langle q|_{AB}\otimes|00\rangle\langle 00|_{E_1E_2}U^{\text{BS}\dagger}_{AE_1}U^{\text{BS}\dagger}_{BE_2}\right), \tag{6}$$

where $|0\rangle$ is the vacuum state and $U^{\text{BS}}_{12} = \exp[i\eta(\hat{a}^\dagger_1\hat{a}_2 + \hat{a}_1\hat{a}^\dagger_2)]$ is the beam splitter operator on systems 1 and 2 with the reflectance $r = \sin\eta$ and the transmittance $t = \cos\eta$. The two-mode squeezed channel can then be expressed as the following density matrix in a photon number basis:

$$\Phi_{\text{channel}} = (1-q^2)\sum_{n,m}\sum_{\substack{k,l=\\ \max\{0,n-m\}}}^{n} c_{nmkl}q^{n+m}t^{2(m-n)+2(k+l)}r^{4n-2(k+l)} \times|kl\rangle\langle m-n+k,m-n+l|, \tag{7}$$

where c_{nmkl} is a coefficient given as

$$c_{nmkl} = \sqrt{\binom{n}{k}\binom{n}{l}\binom{m}{n-k}\binom{m}{n-l}} \tag{8}$$

with binomial coefficients $\binom{n}{k} = n!/[k!(n-k)!]$.

If one tries to teleport an input state $|\psi\rangle$ using the same process for the lossless case, one obtains the output state

$$\rho_{B,\text{out}}(\beta) = \hat{D}_B(g\beta)\langle\beta|_{AR}(|\psi\rangle\langle\psi|_A\otimes\Phi_{RB,\text{channel}})|\beta\rangle_{AR}\hat{D}_B. \tag{9}$$

Using this result, one can adapt the teleportation operator formulation to this lossy environment to define the lossy teleportation operator, which depends on the measurement outcome β and the photon loss number on each mode, k and l as follows:

$$\hat{T}_{kl}(\beta) = \sqrt{\frac{1-q^2}{\pi}}\sum_{n=\max(k,l)}^{\infty}q^n\sqrt{\binom{n}{k}\binom{n}{l}}t^{2n-(k+l)}r^{k+l} \times\hat{D}(g\beta)|n-l\rangle\langle n-k|D(-\beta), \tag{10}$$

so that the output state can be expressed concisely as

$$\rho_{\text{out}}(\beta) = \sum_{k,l=0}^{\infty}\hat{T}_{kl}(\beta)|\psi\rangle\langle\psi|\hat{T}^\dagger_{kl}(\beta). \tag{11}$$

One can easily check that this lossy teleportation operator is a Kraus operator by showing that

$$\sum_{k,l=0}^{\infty} \int \mathrm{d}^2\beta \; \hat{T}_{kl}^{\dagger}(\beta)\hat{T}_{kl}(\beta) = 1. \quad (12)$$

B. Teleportation Fidelity

This approach has the advantage that it simplifies the calculation of the average transmission fidelity by enabling one to carry out an integration for each photon loss number case [corresponding to each different pair of (k,l)] and simply sum them as

$$F = \sum_{k,l} F_{kl} = \sum_{k,l} \int \mathrm{d}^2\beta |\langle\psi|\hat{T}_{kl}(\beta)|\psi\rangle|^2, \quad (13)$$

where $|\psi\rangle$ is the input state of teleportation.

Using the closed form derived in Appendix A, one can calculate the transmission fidelity of the simplest case of the vacuum state $|\psi\rangle = |0\rangle$ as

$$F_{0\to 0} = \frac{1-q^2}{(1+g^2)(1-q^2r^2) - 2gq(1-r^2)}. \quad (14)$$

We note that the lossless fidelity derived in Ref. [26] is recovered when $r = 0$ as follows:

$$F_{0\to 0}^{\text{perfect}} = \frac{1-q^2}{1-2gq+g^2}. \quad (15)$$

The transmission fidelity of the one-photon state ($|\psi\rangle = |1\rangle$) is more complicated, but it can still be expressed in a closed form

$$\begin{aligned} F_{1\to 1} = {} & \frac{1-q^2}{(1-q^2r^4)^2[(1+g^2)(1-q^2r^2) - 2gqt^2]^3} \\ & \times \{qt^2\{qr^2[(g-q)^2 + (1-gq)^2](3+q^2r^4) \\ & + 2(g-q)(1-gq)(1+3q^2r^4)\}[(1+g)^2 \\ & \times (1-q^2r^2) - 2gqt^2] \\ & + q^2t^4(1+q^2r^4)[(1+g^2)(1-q^2r^2) - 2gqt^2]^2 \\ & + 2[1-gq+qr^2(g-q)]^2[g-q+qr^2(1-gq)]^2\}, \end{aligned} \quad (16)$$

where $t^2 = 1 - r^2$. We note again that the lossless transmission fidelity can be obtained by setting $r = 0$ as follows:

$$F_{1\to 1}^{\text{perfect}} = \frac{1-q^2}{(1-2gq+g^2)^3}[(g-q)^2(1-gq)^2 + g^2(1-q^2)^2]. \quad (17)$$

Now, if one tries to teleport a horizontally polarized input state, which can be written as a product state in two orthogonal polarization modes $|H\rangle = |1\rangle_H|0\rangle_V$, one can implement the teleportation for this dual rail qubit by simply running the single-mode teleportation process for each mode [26]. Of course, this requires the same setting for the two teleportation processes, but this assumption is acceptable when the two processes are implemented successively in a short period of time. Since it is equivalent to carry out two independent teleportation processes, the overall average fidelity is simply a product of each mode's fidelity

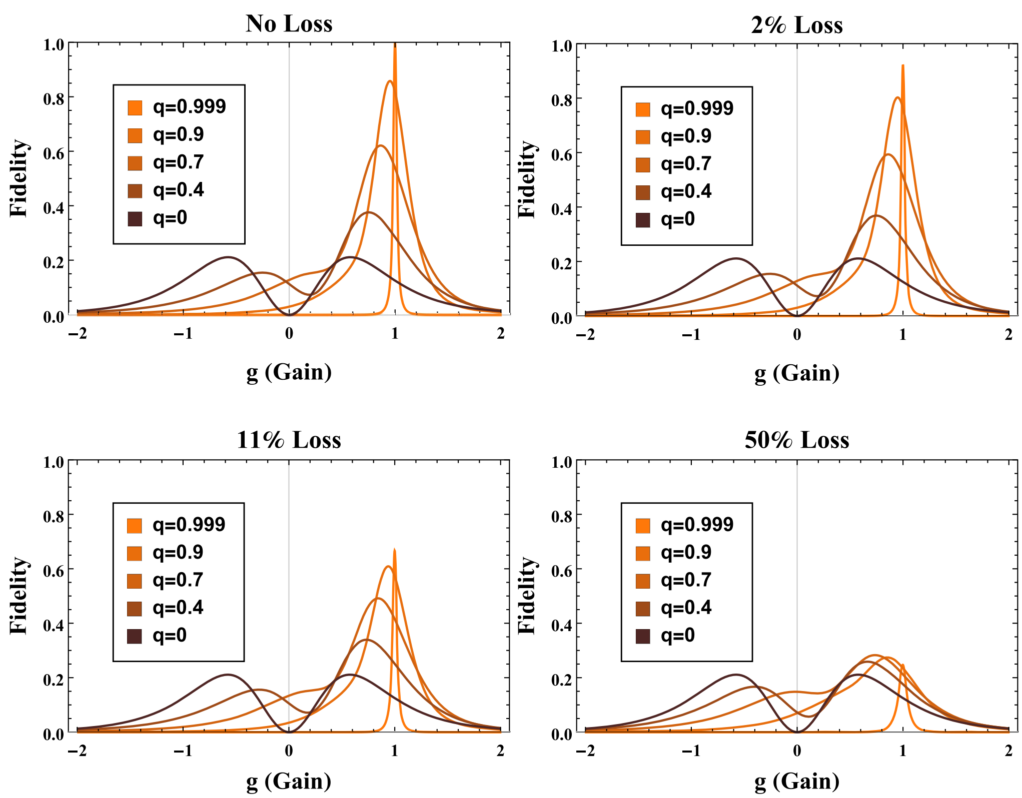

Fig. 1. Fidelity change over the gain value g between −2 and 2 with a varying amount of losses in the two-mode squeezed state.

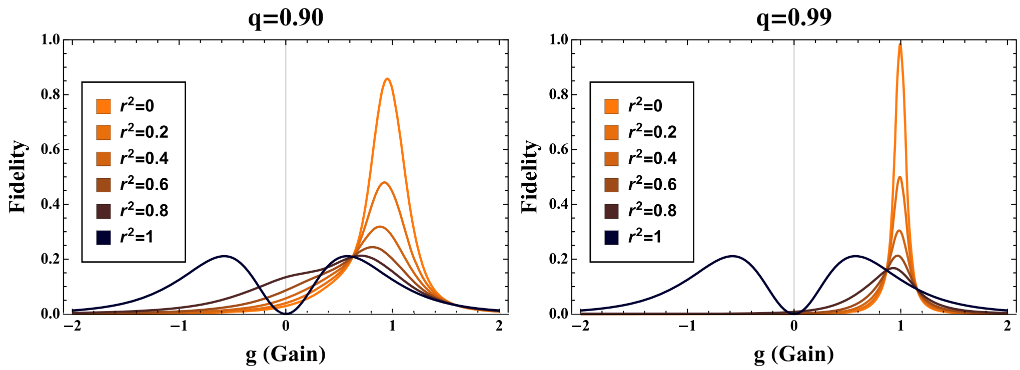

Fig. 2. Fidelity curves over the gain value g between −2 and 2 changing as the loss rate r^2 increases.

$$F_{\text{qubit}} = F_{0\to0} F_{1\to1}. \tag{18}$$

Some results are plotted in Figs. 1 and 2. However, as we show in Appendix B, the overall fidelity F_{qubit} does not depend on the polarization of the input state, so we can see that Eq. (18) is the average transmission fidelity for an arbitrary single-photon input. This rather interesting property may be understood from the comparison with the typical quantum teleportation scheme for a qubit. In the usual scheme, the sender sends two bits of classical information generated from a Bell measurement that can be interpreted as encoding amplitude and phase information for a given basis. The choice of the basis is, however, independent from the eigenbasis of the given input. A pair of complex numbers β generated from the pair of independent CV teleportation protocols is analogous to the pair of bits in the quantum teleportation scheme for a qubit.

Figure 3 shows the average overall fidelity F_{qubit} with optimal gain g_{opt} over the loss expressed as the reflectivity r^2 of the beam splitter modelling the photon loss of the channel for each squeezing parameter q. In the region of $F_{\text{qubit}} > 2/3$, where the fidelity is higher than the limit of the classical teleportation so that the quantum teleportation is meaningful, the stronger the initial squeezing of the channel, the higher the average fidelity. This seems at first to contradict the previous result of Ref. [21], that stronger initial squeezing makes the channel more vulnerable to the thermal loss. However, one can still observe that the inversion in order indeed happens around the region of $F_{\text{qubit}} \approx 0.35$, although it is practically meaningless since it happens in the region where classical teleportation is better than quantum teleportation.

It is interesting to compare this result with the behavior of the log negativity [27] of the Gaussian state [28], which has good properties such as monotonicity under local operations and classical communications [29] and being an upper bound of the distillable entanglement [30]. The log negativity $E_{\mathcal{N}}$ of a Gaussian state ρ can be obtained as

$$E_{\mathcal{N}}(\rho) = \max\{0, -\ln[\tilde{\nu}_-(\rho)]\}, \tag{19}$$

where $\tilde{\nu}_-(\rho)$ is the smaller symplectic eigenvalue of ρ, and is explicitly given for the state Φ_{channel} as

$$\tilde{\nu}_-(\rho) = (1-\mathcal{R})\cosh(2\,\text{artanh}\,q) + \mathcal{R} - (1-\mathcal{R})\sinh(2\,\text{artanh}\,q), \tag{20}$$

where $\mathcal{R} = r^2$ is the reflectivity of the beam splitter modelling the photon loss. The log negativity of Φ_{channel} is plotted in Fig. 4.

The figure shows that the higher initial squeezing strictly guarantees the higher entanglement regardless of the amount of photon loss in the channel, although by no means is the log negativity the unique measure of entanglement. This result is obviously different from the cases of quantum macroscopicity

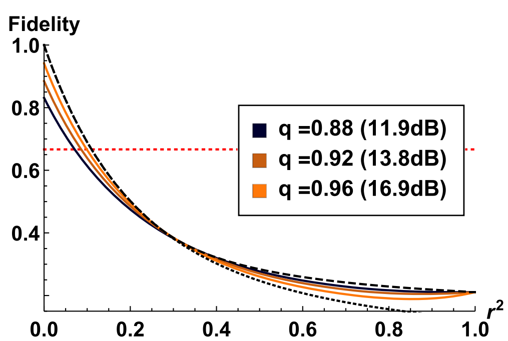

Fig. 3. Transmission fidelity of a photonic qubit with numerically optimized gains. Here, q is the squeezing parameter of the initial two-mode squeezed state; the maximum fidelity curve is numerically obtained and plotted with the thick dashed curve, and that of $q = 1$ is plotted with the thin dotted curve. It is impossible to attain fidelity higher than the classical limit (horizontal dotted line) with loss bigger than about 11%.

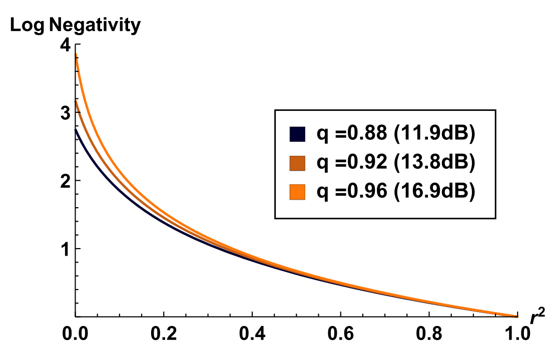

Fig. 4. Log negativity of the two-mode squeezed state in terms of the loss rate r^2. On the contrary to the fidelity, there is no inversion in order with increasing loss.

and Bell-type nonlocality for Gaussian CV states that are destroyed faster under lossy environments when their initial squeezing is larger. It suggests that the vulnerability of performance of tasks exploiting the nonlocality of the two-mode squeezed state with strong squeezing is not because of its rapid decay of entanglement.

In the aspect of the implementation of the teleportation scheme, in the region where the fidelity is higher than the classical limit, 2/3, one may be able to compensate for the detrimental effect from photon loss to some extent by increasing the initial squeezing. However, there is a fundamental upper bound to this as shown in Fig. 3. In the region of $F_{\mathrm{qubit}} > 2/3$ with nonzero loss, even infinite initial squeezing does not allow unit fidelity. One can explicitly calculate this upper bound as a function of the reflectance $\mathcal{R}$ by substituting q and g with 1 in Eq. (18) as

$$F_{q=g\to 1}(\mathcal{R}) = \frac{1+\mathcal{R}^2}{(1+\mathcal{R})^4}. \tag{21}$$

By solving the equation $F_{q=g\to 1}(\mathcal{R}) = 2/3$, one can find the biggest loss $\mathcal{R}_{\mathrm{classic}}$ with which one can attain fidelity higher than the classical limit as

$$\mathcal{R}_{\mathrm{classic}} \approx 0.11. \tag{22}$$

This implies that even if the loss rate of the optical cable for the two-mode squeezed state is 0.1 dB/km, after traveling more than 12 km, it cannot be used to implement the teleportation scheme. It should be noted that this limitation cannot be circumvented by increasing the initial entanglement, since this bound is attained with the infinitely entangled initial state.

There have been studies on photon loss thresholds that can be tolerated by various all-optical quantum computing schemes [6,7,31–34]. Our result implies that the hybrid scheme can be valid for fault-tolerant quantum computing only when the photon loss rate can be suppressed under a certain limit. In order to obtain 99% (99.8%) fidelity for teleportation with infinite squeezing, the photon loss rate should be under 0.252% (0.050%). Of course, it is extremely challenging to generate a pure squeezed state without noise and large squeezing over 13 dB [35]. We may consider reasonable levels of squeezing for practical realizations in the foreseeable future. For example, with 20 dB squeezing, the photon loss rate should be under 0.6% to obtain 95% fidelity for teleportation, and the maximum fidelity without loss is only 97%.

One of the side effects when deploying the CV teleportation protocol for a qubit is the leakage problem, i.e., the presence of nonzero amplitudes of multiphoton terms in the output state of teleportation. One may circumvent this problem by projecting the output state onto the qubit subspace, i.e., the subspace of the Fock space spanned by $|0\rangle$ and $|1\rangle$ [19]. However, this contradicts the merit of the hybrid teleportation scheme that it is deterministic [19], which is considered an advantage compared to the nondeterministic Bell-measurement-based teleportation [5]. Moreover, such post-selection requires a nontrivial quantum nondemolition measurement that filters out multiphoton states (i.e., $|2\rangle, |3\rangle, \ldots$), while it leaves zero- or one-photon states untouched. This implies that although it is possible to detect the leakage error, its correction is hard.

Even though this technical problem might be overcome, a high teleportation fidelity (for example, around 0.9 with 20% photon loss after an appropriate post-selection [23]) could be obtained only at the cost of a formidably low success probability (up to 60%, even with a lossless squeezed state resource with 17 dB squeezing) that is far from being “nearly deterministic” [23]. Such figures may be deemed misleading considering the purpose of the hybrid teleportation scheme. In this sense, the average fidelity calculated here has its own operational meaning as the output fidelity without any filtering or post-selection, in comparison to the hypothetical post-selected output fidelity.

4. REMARKS

The schemes for all-optical quantum computation using photonic qubits typically suffer from limited fault-tolerance limits due to the nondeterministic nature of gate teleportation operations [5,7]. As an alternative approach, a hybrid teleportation scheme that is to teleport a photonic qubit via a continuous-variable channel has been suggested and experimentally demonstrated [19]. In this way, a qubit can be teleported deterministically, and the fidelity can be made arbitrarily high by increasing the degree of squeezing for the continuous-variable channel. It is, however, also true that a small amount of photon loss may significantly degrade the teleportation fidelity in the limit of large squeezing because largely squeezed states have essential properties of macroscopic superpositions [21]. We thus find the upper bound of the fidelity when teleporting a photonic qubit via a two-mode squeezed channel under a lossy environment. Indeed, when squeezing is large, the teleportation fidelity rapidly degrades. The increase of photon loss decreases this bound, and teleportation better than this limit is impossible even when the squeezing degree of the teleportation channel becomes infinity. Our result indicates that the hybrid scheme can be valid for fault-tolerant quantum computing only when the photon loss rate can be suppressed under a certain limit.

APPENDIX A: DERIVATION OF THE CLOSED FORM OF FIDELITY

For more general purposes, we derive a closed form of the following series expression:

$$\sum_{n,m=0}^{\infty} \sum_{k,l=0}^{\min(n,m)} \binom{n}{k}\binom{m}{l}\binom{n+m-k-l}{m-k} A^n B^m C^k D^l. \tag{A1}$$

One can observe that this expression immediately gives the fidelity of the vacuum state by the substitution

$$A = B = \frac{gqt^2}{1+g^2}, \qquad C = \left(\frac{r}{t}\right)^2 (1+g^2), \qquad D = \frac{B}{g^2}. \tag{A2}$$

Now Eq. (A1) can be rewritten as an integral expression

$$\frac{1}{\pi}\int_0^{\infty} \mathrm{d}y \int_{\mathbb{C}} \mathrm{d}^2 x e^{-y} e^{B(\sqrt{y}+\sqrt{A}Cx)(\sqrt{y}+\sqrt{A}Dx^*)} e^{\sqrt{Ay}(x+x^*)} e^{-|x|^2}, \tag{A3}$$

which includes a simple Gaussian integration with respect to the variable x. This integral expression yields a closed form as follows:

$$\frac{1}{1-A-B-A^2C-B^2D-ABCD}. \quad \text{(A4)}$$

Now the transmission fidelity of one photon can be expressed in the integral form

$$F_{1\to 1} = \frac{(1-q^2)(1+g^2)}{\pi g^2}\int_0^\infty \mathrm{d}y \int C\mathrm{d}^2x[f(x,x^*,y) \times e^{-y} e^{A(\sqrt{y}+\sqrt{A}Bx)(\sqrt{y}+\sqrt{A}Cx^*)} e^{\sqrt{Ay}(x+x^*)} e^{-|x|^2}], \quad \text{(A5)}$$

where

$$\begin{aligned} f(x,x^*,y) = {} & A\left(\sqrt{A}x - \frac{\sqrt{y}}{1+g^2}\right)\left(\sqrt{A}x^* - \frac{g^2\sqrt{y}}{1+g^2}\right) \\ & + \left[A(\sqrt{y}+\sqrt{A}Bx) - \frac{\sqrt{y}}{1+g^2}\right] \\ & \times \left[A\left(\sqrt{y}+\sqrt{A}\frac{B}{g^2}x^*\right) - \frac{g^2\sqrt{y}}{1+g^2}\right] \\ & \times \left(\sqrt{A}x - \frac{\sqrt{y}}{1+g^2}\right)\left(\sqrt{A}x^* - \frac{g^2\sqrt{y}}{1+g^2}\right). \end{aligned} \quad \text{(A6)}$$

By using the following relation repeatedly in the calculation of the integral expression, one can also obtain a closed form of the fidelity $F_{1\to 1}$:

$$\begin{aligned} & \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2x\, x^n x^{*m} \exp(-a|x|^2 + b_1 x + b_2 x^*) \\ & = \frac{1}{a}\frac{\partial^n}{\partial b_1^n}\frac{\partial^m}{\partial b_2^m} b_2 \exp\left(\frac{b_1 b_2}{a}\right). \end{aligned} \quad \text{(A7)}$$

APPENDIX B: POLARIZATION INDEPENDENCE OF THE TRANSMISSION FIDELITY IN A LOSSY CHANNEL

A proof of this fact was first given in Ref. [23], but the proof given here follows the logic of the one given in Ref. [26] adapted for the lossy channel. The term "polarization" used here does not necessarily mean the polarization in two independent oscillation modes of the electric field but is used to refer to coherence between two arbitrarily chosen modes sharing one photon between them. In other words, we chose a state

$$|H\rangle = |1\rangle_H|0\rangle_V \quad \text{(B1)}$$

with its complementary orthogonal state $|V\rangle = |0\rangle_H|1\rangle_V$ as an input state in the main article, but one may also use

$$|S\rangle = c_H|H\rangle + c_V|V\rangle, \quad \text{(B2)}$$

$$c_H^2 + c_V^2 = 1, \quad \text{(B3)}$$

where we can set c_H as a real number since the global phase factor has no physical significance, and also c_V, because if c_V is not real, then one can define a new $|V\rangle = \frac{c_V}{|c_v|}|0\rangle_H|1\rangle_V$. Since we have freedom of choice of the orthogonal complement state to $|S\rangle$, just as $|V\rangle$ was an orthogonal complement state to $|H\rangle$, we define $|P\rangle = c_V|H\rangle - c_H|V\rangle$ as an orthogonal complement state to $|S\rangle$. Let $\hat{U}$ be a unitary transformation between two modes as follows:

$$\begin{aligned} \hat{U}(a_H^\dagger \otimes 1_V)\hat{U}^\dagger &= a_S^\dagger \otimes 1_P, \\ \hat{U}(1_H \otimes a_V^\dagger)\hat{U}^\dagger &= 1_S \otimes a_P^\dagger. \end{aligned} \quad \text{(B4)}$$

$\hat{U}$ defined in this way has a property that it has the same effects on $\hat{a}_H^\dagger$ and $\hat{a}_V^\dagger$ with $\hat{U}^\dagger$ as

$$\begin{aligned} \hat{U}^\dagger(a_H^\dagger \otimes \mathbb{1}_V)\hat{U} &= a_S^\dagger \otimes \mathbb{1}_P, \\ \hat{U}^\dagger(\mathbb{1}_H \otimes a_V^\dagger)\hat{U} &= \mathbb{1}_S \otimes a_P^\dagger. \end{aligned} \quad \text{(B5)}$$

The fidelity for the $|S\rangle$ input can be written as

$$F_{\text{qubit}} = \sum_{kl,mn}\int \mathrm{d}^2\beta_H \mathrm{d}^2\beta_V |\langle H|\hat{U}^\dagger \hat{T}_{kl,mn}(\beta_H,\beta_V)\hat{U}|H\rangle|^2, \quad \text{(B6)}$$

where $\hat{T}_{kl,mn}(\beta_H,\beta_V) = \hat{T}_{kl}(\beta_H)\otimes\hat{T}_{mn}(\beta_V)$. Now following the original proof, we identify the teleportation operator $\hat{T}_{kl}(\beta)$ for $\beta = 0$ as follows:

$$\sqrt{\frac{1-q^2}{\pi}}\frac{\left(\frac{r}{t}\hat{a}\right)^k}{\sqrt{k!}}(t^2q)^{\hat{n}}\frac{\left(\frac{r}{t}\hat{a}^\dagger\right)^l}{\sqrt{l!}}, \quad \text{(B7)}$$

where $f(\hat{n})$ is understood as $\sum_{n=0}^\infty f(n)|n\rangle\langle n|$. The connection to the teleportation operator for the lossless case can be found as

$$\hat{T}_{kl}(\beta) = \frac{\left[\frac{r}{t}(\hat{a}+g\beta)\right]^k}{\sqrt{k!}}\frac{\left[rtq(\hat{a}^\dagger+g\beta^*)\right]^l}{\sqrt{l!}}\hat{T}^g_{t^2q}(\beta). \quad \text{(B8)}$$

Now we rewrite the fidelity in terms of superoperators as

$$\begin{aligned} F_{\text{qubit}} = \int & \mathrm{d}^2\beta_H \mathrm{d}^2\beta_V \langle H|\mathcal{U}^\dagger \circ e^{\frac{r}{t}[\mathcal{E}_{H-}(g\beta_H)+\mathcal{E}_{V-}(g\beta_V)]} \\ & \circ e^{rtq[\mathcal{E}_{H+}(g\beta_H)+\mathcal{E}_{V+}(g\beta_V)]} \circ \mathcal{T}^g_{t^2q}(\beta_H,\beta_V) \\ & \circ \mathcal{U}(|H\rangle\langle H|)|H\rangle, \end{aligned} \quad \text{(B9)}$$

where $\mathcal{U}(\rho) = U\rho U^\dagger$ is the polarization rotation unitary transform, $\mathcal{E}_{X+}(\alpha)(\rho) = (\hat{a}_X^\dagger + \alpha^*)\rho(\hat{a}_X + \alpha)$ and $\mathcal{E}_{X-}(\alpha)(\rho) = (\hat{a}_X + \alpha)\rho(\hat{a}_X + \alpha^*)$ are displaced photon creation/annihilation operations, and $\mathcal{T}^g_q(\beta_H,\beta_V)(\rho) = \hat{T}^g_q(\beta_H)\otimes\hat{T}^g_q(\beta_V)\rho \cdot \hat{T}^g_q(\beta_H)^\dagger \otimes \hat{T}^g_q(\beta_V)^\dagger$ is the lossless teleportation operator. From Ref. [26], it is known that $\mathcal{U}^\dagger \circ \mathcal{T}^g_q(\beta_H,\beta_V)\circ\mathcal{U} = \mathcal{T}^g_q(\beta_S,\beta_P)$, where

$$\begin{aligned} \beta_S &= c_H\beta_H + c_V\beta_V, \\ \beta_P &= c_V\beta_H - c_H\beta_V. \end{aligned} \quad \text{(B10)}$$

Now we are going to prove that

$$\mathcal{U}^\dagger \circ e^{a[\mathcal{E}_{H\pm}(\beta_H)+\mathcal{E}_{V\pm}(\beta_V)]} \circ \mathcal{U} = e^{a[\mathcal{E}_{H\pm}(\beta_S)+\mathcal{E}_{V\pm}(\beta_P)]}. \quad \text{(B11)}$$

To show this, it is sufficient to prove that

$$\mathcal{U}^\dagger \circ [\mathcal{E}_{H\pm}(\beta_H)+\mathcal{E}_{V\pm}(\beta_V)]\circ\mathcal{U} = [\mathcal{E}_{H\pm}(\beta_S)+\mathcal{E}_{V\pm}(\beta_P)], \quad \text{(B12)}$$

which holds because for any state ρ,

$$
\begin{aligned}
&\{\mathcal{U}^\dagger \circ [\mathcal{E}_{H+}(\beta_H) + \mathcal{E}_{V+}(\beta_V)] \circ \mathcal{U}\}(\rho) \\
&\quad = U^\dagger[(\hat{a}_H^\dagger + \beta_H^*)U\rho\, U^\dagger(\hat{a}_H + \beta_H) + (\hat{a}_V^\dagger + \beta_V^*) \\
&\qquad \times U\rho\, U^\dagger(\hat{a}_V + \beta_V)]U \\
&\quad = (c_H\hat{a}_H^\dagger + c_V\hat{a}_V^\dagger + \beta_H^*)\rho(c_H\hat{a}_H + c_V\hat{a}_V + \beta_H) \\
&\qquad + (c_V\hat{a}_H^\dagger - c_H\hat{a}_V^\dagger + \beta_V^*)\rho(c_V\hat{a}_H - c_H\hat{a}_V + \beta_V) \\
&\quad = (\hat{a}_H^\dagger + c_H\beta_H^* + c_V\beta_V^*)\rho(\hat{a}_H + c_H\beta_H + c_V\beta_V) \\
&\qquad + (\hat{a}_V^\dagger + c_V\beta_H^* - c_H\beta_V^*)\rho(\hat{a}_V + c_V\beta_H - c_H\beta_V) \\
&\quad = [\mathcal{E}_{H+}(\beta_S) + \mathcal{E}_{V+}(\beta_P)](\rho),
\end{aligned} \tag{B13}
$$

and a similar equation holds for the case of $\mathcal{E}_{H-}(\beta_H) + \mathcal{E}_{V-}(\beta_V)$. Now we can rewrite Eq. (B9) as

$$
\begin{aligned}
F_{\text{qubit}} = \int \mathrm{d}^2\beta_H \mathrm{d}^2\beta_V \langle H| e^{\frac{r}{t}[\mathcal{E}_{H-}(g\beta_S)+\mathcal{E}_{V-}(g\beta_P)]} \\
\circ\, e^{rtq[\mathcal{E}_{H+}(g\beta_S)+\mathcal{E}_{V+}(g\beta_P)]} \circ \mathcal{T}^{g}_{t^2 q}(\beta_S, \beta_P)(|H\rangle\langle H|)|H\rangle.
\end{aligned} \tag{B14}
$$

However, by changing the integration variables from $\{\beta_H, \beta_V\}$ to $\{\beta_S, \beta_P\}$, one can see that it is the transmission fidelity of state $|H\rangle$.

Funding. National Research Foundation of Korea (NRF) (2010-0018295); Korea Institute of Science and Technology (KIST) (2E27800-18-P043).

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