Classical Mechanics II (Fall 2020): Homework #4

Due Dec. 3, 2020

[0.5 pt each, total 6 pts]

1. Thornton & Marion, Problem 13-9

(Note: For Problem 13-9, first review Chapter 13.5 and work out Example 13.2, which was briefly discussed in the class, but left for your exercise.)

• The complementary solution is found by plugging $\beta = \frac{D}{2\rho}$ and $\omega_0^2 = \frac{s^2\pi^2\tau}{\rho b^2}$ of Eq.(13.49) into Eq.(3.54) or Eq.(3.37).

2. Thornton & Marion, Problem 13-19

• $\Psi_1(x,t) = Ae^{-i(k_1 x-\omega t)} + Be^{i(k_1 x+\omega t)}$, $\Psi_2(x,t) = Ce^{-i(k_2 x-\omega t)} + De^{i(k_2 x+\omega t)}$, and $\Psi_3(x,t) = Ee^{-i(k_1 x-\omega t)}$ with boundary conditions $\Psi_1(0,t) = \Psi_2(0,t)$, $\frac{\partial \Psi_1}{\partial x}(0,t) = \frac{\partial \Psi_2}{\partial x}(0,t)$, $\Psi_2(L,t) = \Psi_3(L,t)$, $\frac{\partial \Psi_2}{\partial x}(L,t) = \frac{\partial \Psi_3}{\partial x}(L,t)$, where $\frac{\omega}{k_1} = \sqrt{\frac{\tau}{\rho_1}}$ and $\frac{\omega}{k_2} = \sqrt{\frac{\tau}{\rho_2}}$. To allow a maximum transmission, L must be chosen in such a way that maximizes $\frac{|E|^2}{|A|^2}$ and minimizes $\frac{|B|^2}{|A|^2}$.

3. Thornton & Marion, Problem 13-22

• $\Psi(x,0) = B\sqrt{\frac{\pi}{\sigma}}\, e^{-x^2/4\sigma} e^{-ik_0 x}$ and $\Psi(x,t) = B\sqrt{\frac{\pi}{\sigma}}\, e^{-(\omega_0' t - x)^2/4\sigma} e^{i(\omega_0 t - k_0 x)}$ listed in the problem have the same $1/e$ width of $W_{1/e}(t) = 2 \times 2\sqrt{\sigma} =$ constant. However, with the first three terms of the Taylor expansion of $\omega(k)$ in Eq.(13.113), one finds $W_{1/e}(t) = 4\sqrt{\sigma}\left[1 + \left(\frac{\omega_0'' t}{2\sigma}\right)^2\right]^{\frac{1}{2}}$ which increases in time.

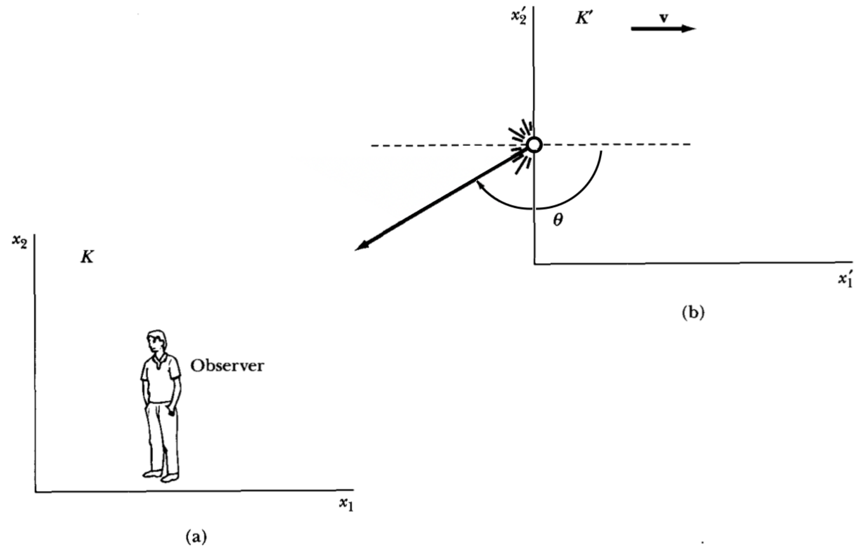
4. Thornton & Marion, Problem 14-6

(Note: In Problem 14-6, as in other examples and problems, unprimed quantities such as Δx is measured in the unprimed system K.)

• Using the transformation Eq.(14.14) from K to K' frame, $t_1' = \gamma\left(t_1 - \frac{vx_1}{c^2}\right)$ and $t_2' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right) \;\rightarrow\; \Delta t' = t_1' - t_2' = \gamma\left[(t_1 - t_2) - \frac{v(x_1 - x_2)}{c^2}\right] = -\gamma\frac{v\Delta x}{c^2}$.

5. Thornton & Marion, Problem 14-18

(Note: For Problem 14-18, consider the choice of systems below, in which the angle between the light source and the direction of the relative motion is again θ as in Example 14.11.)



• Unlike Example 14.11, assume now that the observer is in system K and the source is in system K' moving at speed v with respect to K. Using the transformation Eq.(14.92) from K' to K frame, $E = \gamma(E' + vp_1') = \gamma(h\nu_0 + v\frac{h\nu_0}{c}\cos\theta) = h\nu \quad \rightarrow \quad \nu = \gamma\nu_0(1 + \beta\cos\theta)$, where ν is now the desired observed photon frequency. Thus, when the source approaches the observer ($\theta = 0$), $\nu = \nu_0\sqrt{\frac{1+\beta}{1-\beta}}$, and when the source recedes from the observer ($\theta = \pi$), $\nu = \nu_0\sqrt{\frac{1-\beta}{1+\beta}}$.

6. Thornton & Marion, Problem 14-33

• Adopting the notations in Figure 14-9, in the CM (Center of Momentum) frame, the momentum conservation states $0 = \mathbf{p}_p' + \mathbf{p}_e' + \mathbf{p}_{\bar{\nu}}'$ (from which one easily finds $|\mathbf{p}_p'| = |\mathbf{p}_e'| = |\mathbf{p}_{\bar{\nu}}'| \equiv p'$), and the energy conservation states $m_n c^2 = \sqrt{m_p^2 c^4 + p'^2 c^2} + \sqrt{m_e^2 c^4 + p'^2 c^2} + p'c\,.$

7. Thornton & Marion, Problem 14-37

• The Lagrange's equation of motion is found to be $\frac{mc^2\beta d\beta}{(1-\beta^2)^{3/2}} + kx dx = 0$, for which we use $c\dot{\beta} = c\beta^2\frac{d\beta}{dx}$.

8. Thornton & Marion, Problem 14-42

• From energy conservation, $h\nu + m_e c^2 = h\nu' + \sqrt{p_e^2 c^2 + m_e^2 c^4}$, and momentum conservation, $p_e^2 = \left(\frac{h\nu}{c}\right)^2 + \left(\frac{h\nu'}{c}\right)^2 - 2\left(\frac{h\nu}{c}\right)\left(\frac{h\nu'}{c}\right)\cos\theta$, you can eliminate p_e to acquire the familiar Compton scattering formula, $\frac{c}{\nu'} - \frac{c}{\nu} = \frac{h}{m_e c}(1 - \cos\theta)$. By dividing by $\frac{c}{\nu}$ and inverting both sides, we can easily derive $\frac{E'}{E} = \left[1 + \frac{E}{m_e c^2}(1 - \cos\theta)\right]^{-1}$.

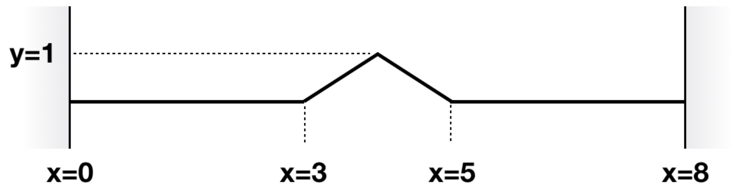
9. Consider a triangular pulse on a finite, dissipationless string discussed in the class. Here we will investigate the general traveling wave solution to the wave equation.

(a) First, let us re-examine the plucked string described in Figure 13-1 and Example 13.1. Using a trigonometric identity $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$, show that Eq.(13.13) can be written in the form of $\frac{1}{2}[f(x+vt) + f(x-vt)]$, or Eq.(13.62b). Find $f(X)$.

(b) By a numerical calculation using e.g., the first 5 nonzero terms of the series to get a moderate approximation, draw the shapes of *(i)* $\frac{1}{2}f(x+vt)$, *(ii)* $\frac{1}{2}f(x-vt)$, and *(iii)* the combined actual string shape. You may plot the position of the string at e.g., 8 equally spaced times from $t = 0$ to $t = T$, where T is the period of the string's motion. You may certainly reuse what you developed for Problem 13-3 in Homework #3.

(c) Now consider a slightly different initial configuration. A string of length $L = 8$ is fixed at both ends, and is initially given a small triangular displacement shown in the figure below. The string is then released from rest at $t = 0$. Describe the vibration of the string in terms of normal modes. Find the Fourier coefficients, Eq.(13.8).

(d) By a numerical calculation using e.g., the first 5 nonzero terms of the series to get a moderate approximation, see how the wave propagates in time. You may plot the position of the string at e.g., 20 equally spaced times from $t = 0$ to $t = T$, where T is the period of the string's motion. What is T? In your first few snapshots, can you reproduce the behavior seen in Figure 13-3?



- (a) $f(X) = \sum_r \mu_r \sin\left(\frac{r\pi X}{L}\right)$ where μ_r is given in Example 13.1.

- (c) $\nu_r = 0$ and $\mu_{2r'} = 0$ where r' is a non-negative integer, but,

$$\mu_{2r'+1} = \frac{32 \cdot (-1)^{r'}}{(2r'+1)^2\pi^2}\left(1 - \cos\frac{(2r'+1)\pi}{8}\right).$$

- (d) For the triangular pulse to return to its original configuration of displacement, it takes two reflections for each of the two triangular waves going left and right.

10. A particle (particle 1) of mass m and relativistic total energy E_1 collides with an identical particle (particle 2) initially at rest. The particles collide to reach a final state containing N particles of mass m each (e.g., a collection of particles and anti-particles, all with mass m). Find the *threshold* energy, or the minimum value of E_1, for which this process can occur. Show that more energy would be needed to produce more particles.

- Adopting the notations in Figure 14-9, in the CM (Center of Momentum) frame the energy conservation states $E'_1 + E'_2 = 2E'_1 = Nmc^2$. From $E'_1 = \frac{N}{2}mc^2$ we can tell $\gamma'_1 = \gamma'_2 = \frac{N}{2}$ for both particles in the CM frame. The speed of both particles in the CM frame is the same as the relative speed between the CM and LAB frame, i.e. $\gamma'_1 = \gamma'_2 = \gamma$ or $u'_1 = u'_2 = V = \frac{u_1}{2}$.

• Then, using the transformation Eq.(14.92) but from K' to K frame, $E_1 = \gamma(E_1' + Vp_1') = \gamma(\frac{N}{2}mc^2 + V\gamma_1 mu_1') = \gamma(\frac{N}{2}mc^2 + \gamma mV^2) = \left(\frac{N^2}{2} - 1\right)mc^2$, where we used $V^2 = \left(1 - \frac{4}{N^2}\right)c^2$ from $\gamma = \frac{N}{2}$.

• Alternatively, one can utilize the fact that the square of the four-vector momentum, Eq.(14.95), is invariant in Lorentz transformation. In the LAB frame, the *total* four-vector momentum of the 2-particle *initial* state is $\left(\sqrt{\frac{E_1^2}{c^2} - m^2c^2}, 0, 0, i\left(\frac{E_1}{c} + mc\right)\right)$. In the CM frame, the *total* four-vector momentum of the N-particle *final* state is $(0, 0, 0, iNmc)$. Equating the squares of the two four-vector momenta yields $-2m^2c^2 - 2E_1 m = -N^2m^2c^2 \quad \rightarrow \quad E_1 = \left(\frac{N^2}{2} - 1\right)mc^2$.

11. A spaceship is initially at rest in the LAB frame. At a given instant (LAB clock $t' = 0$ and the spaceship's clock $t = 0$), it starts to accelerate with the constant *proper* acceleration a along the x_1'- and x_1-axes. Here, the *proper* acceleration is defined as follows: Let t be the time coordinate in the spaceship's frame. If the proper acceleration is a, then at time $t + dt$, the spaceship is moving at speed adt relative to the frame it was in at time t.

(a) Show that the relative speed of the spaceship and the LAB frame at the spaceship's time t is written as $v(t) = c\tanh\left(\frac{at}{c}\right)$. (Note: You may start by setting $v_1 = v(t)$ and $v_2 = adt$ in the velocity addition formula Eq.(14.98) of Thornton & Marion, and expand $v(t + dt)$ to first order in dt.)

(b) Later on, an observer in the LAB frame measures t' and t. Find the relation between t' and t? Check if your formula yields $t' = t$ in the nonrelativistic limit.

(c) Find the four-vector velocities $\mathbb{V}'$ and $\mathbb{V}$ of the spaceship in the LAB frame and the spaceship's frame, respectively. Show that they transform like four-vectors between the two frames.

• (a) Integrating $dv = a\left(1 - \frac{v^2}{c^2}\right)dt$, you get $\frac{v(t)}{c} = \tanh\left(\frac{at}{c}\right)$.

• (b) The stationary observer in the LAB frame will find the clock on the spaceship slows down. Integrating $dt' = \gamma dt$ with $\gamma = \frac{1}{\sqrt{1 - v^2(t)/c^2}} = \cosh\left(\frac{at}{c}\right)$, you get $\frac{at'}{c} = \sinh\left(\frac{at}{c}\right)$.

• (c) Using the four-vector velocity definition Eq.(14.85), you get $\mathbb{V} = (0, 0, 0, ic)$ and $\mathbb{V}' = \gamma(v, 0, 0, ic) = \left(c\sinh\left(\frac{at}{c}\right), 0, 0, ic\cosh\left(\frac{at}{c}\right)\right)$. Comparing our definition of primed and unprimed coordinates with Eq.(14.14), we find $\mathbb{V} = \boldsymbol{\lambda}\mathbb{V}'$ with $\boldsymbol{\lambda}$ as defined in Eq.(14.77).

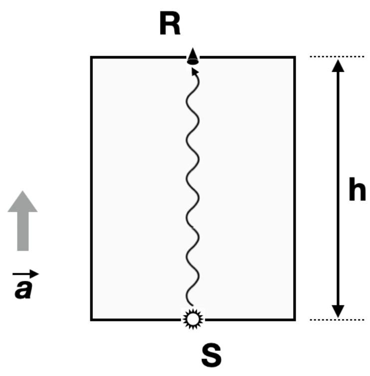
12. To understand the gravitational shift of spectral lines, let us consider an elevator of height h starting to move upward from rest at $t = 0$ with an acceleration $a = g = 9.8\,\mathrm{m/s^2}$ in free space (no gravity) with a photon source S on its floor. A photon of frequency ν_0 is emitted at $t = 0$ and travels upward as seen in the figure below.

(a) An observer R fixed to the elevator's ceiling — thus accelerating with the elevator — receives the photon with Doppler-shifted frequency ν. Show that the frequency difference is given by $\Delta\nu = \nu - \nu_0 \simeq -\frac{gh}{c^2}\nu_0$ when we can assume $\sqrt{gh} \ll c$. (Note: You may want to consider two inertial reference frames — K' moving at the same velocity as the elevator when the photon is emitted, and K moving at the same velocity as the elevator when the photon is received — and the transformation of a four-vector momentum between K' and K.)

(b) Combining the Equivalence Principle (what is this?) and the result of the above thought experiment, explain the *gravitational* redshift (and the *gravitational* time dilation). Notice that, in your gravitational redshift formula, gh is the change in the Newtonian gravitational potential experienced by the photon.

(c) What do you expect if a photon falls from the ceiling rather than moving upward from the floor?

(d) Two people stand a distance h apart. They simultaneously start accelerating in the same direction (along the line connecting the two) with the same *proper* acceleration a. At the instant they start to move, how fast does each person's clock tick when observed by the other person?



• (a) K is receding from K' as K travels faster upward. Hence the photon that K sees (or the observer R receives) will be Doppler-shifted as compared to what K' observes. Either by the Lorentz transformation of the four-vector momentum, Eq.(14.92), or by the relativistic Doppler effect formula, Eq.(14.33) or (14.105), one can show $\frac{\Delta\nu}{\nu_0} = \sqrt{\frac{1-v/c}{1+v/c}} - 1 \simeq -\frac{v}{c}$.

• (b) Neither S or R in the elevator can tell that they are accelerating in deep space or on the surface of the Earth. The result from (a) must hold on the surface of the Earth, as verified by the Pound-Rebka experiment discussed in the class.

• (d) In the usual special relativity situation where two observers fly past each other with relative speed v, they both see the other person's clock slowed down by the same factor. This is because the situation is symmetric between the two observers. However, in this problem the symmetry has been broken, as the acceleration vector determines a special direction in space.

13. [For additional +0.5pt] Einstein's Speed of Light Postulate states that the speed of light c is a universal constant independent of any relative motion of the source and the observer, implying that no information can be transmitted faster than c (i.e., *signal velocity* $\leq c$). Write a short essay that includes your thoughts and research about the following questions: *(i)* Can the *phase velocity* be faster than c, and if so, why does it not contradict the Speed of Light Postulate? *(ii)* Can the *group velocity* be faster than c, and if so, why does it not contradict the Postulate? *(iii)* Can the *velocity on a screen* described in p.576 of Thornton & Marion be faster than c, and if so, why does it not contradict the Postulate?

(Note: 2-3 paragraphs or more are expected to clearly demonstrate what you learned from various scientific articles. You must reference your sources appropriately with a proper citation convention, and your answer must be your own work in your own words. Sources like Wikipedia or YouTube are *not* scientific literatures. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy. You may start by reviewing p.541-542 and p.576 of Thornton & Marion. Other articles that may be a good starting point for your research include: *(i)* Rothman, M. A., 1960, "Things that go faster than light", Scientific American, 203/1, 142-152, *(ii)* Feinberg, G., 1970, "Particles that go faster than light", Scientific American, 222/2, 68-75. Obviously, your literature search should not stop there.)