Classical Mechanics II (Fall 2020): Homework #2

Due Oct. 20, 2020

[0.5 pt each, total 6 pts]

1. Thornton & Marion, Problem 10-8

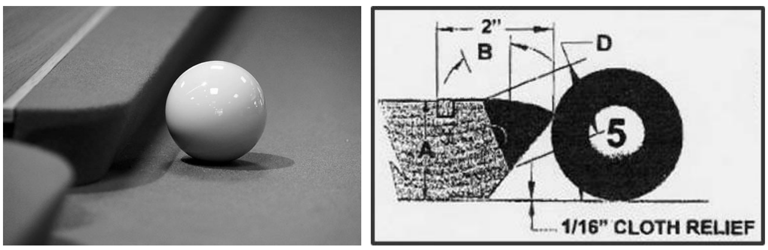
(Note: For Problem 10-8, also sketch the particle's orbit seen from the north — by an observer fixed to the earth — assuming, for simplicity, that it is thrown on the equator. Compare with the orbit described in Example 10.3 in which a particle is dropped from above. Explain why the Coriolis force moves the dropped particle to the east, but the thrown particle to the west.)

• Using the rotating frame in Figure 10-9, $\mathbf{a} = \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{v}_r \rightarrow \ddot{y} = -2\omega \dot{z} \cos\lambda \rightarrow d_y = \omega \cos\lambda \left(\frac{1}{3} g t^3 - v_0 t^2\right)$ with $v_0 = \sqrt{2gh}$ and $t = 2\sqrt{\frac{2h}{g}}$.

2. Thornton & Marion, Problem 10-10

• Recall from the class, $\mathbf{a}_{\rm cor} = -2\boldsymbol{\omega} \times \mathbf{v}_r = (2v_0\omega \sin\lambda \cos\alpha)\,\hat{\mathbf{e}}_x + (-2v_0\omega\cos\lambda\sin\alpha + 2\omega g t \cos\lambda)\,\hat{\mathbf{e}}_y + (2v_0\omega\cos\lambda\cos\alpha)\,\hat{\mathbf{e}}_z \rightarrow$ The exact distance traveled is $y(T') = (v_0 \cos\alpha)T' - (v_0\omega\cos\lambda\sin\alpha)T'^2 + (\omega g \cos\lambda)\cdot \frac{1}{3}T'^3$ where the time of flight is given as $T' = \frac{2v_0 \sin\alpha}{g - 2v_0\omega\cos\lambda\cos\alpha}$.

3. Thornton & Marion, Problem 11-5



Thornton & Marion, Problem 11-5(b), image credits: homeleisuredirect.com, pooltablefeltcloth.com

• $\int \mathbf{F} dt = \mathrm{M}\mathbf{V}_{\rm CM}$ and $\int \boldsymbol{\tau} dt = r \int \mathbf{F} dt = \frac{2}{5}\mathrm{MR}^2 \left(\frac{\mathbf{V}_{\rm CM}}{R}\right) \rightarrow r = \frac{2}{5}\mathrm{R}$.

4. Thornton & Marion, Problem 11-10

• The angular momentum of the system before and after the particle starts to move is $L = \frac{2}{5}MR^2\omega = \left[\frac{2}{5}MR^2 + m(R\sin\theta)^2\right]\dot{\phi}$, where $\dot{\phi}(\theta)$ is the angular velocity when the particle has moved by θ from one pole. Then, the angle by which the system rotated while the particle moved from $\theta = 0$ to π is $\int_0^T \dot{\phi}(\theta)dt = \int_0^T \frac{\frac{2}{5}MR^2}{\frac{2}{5}MR^2 + m(R\sin\theta)^2}dt$.

5. Thornton & Marion, Problem 11-13

(Note: For Problem 11-13, you may opt to use your favorite numerical tool such as MATLAB or MATHEMATICA to e.g., quickly find roots of a characteristic equation of a matrix.)

• From $\det(\mathbf{I} - I\mathbf{1}) = 0$, you acquire, for example, a principal moment of inertia $I_1 = 10mb^2$ with a corresponding principal axis $\boldsymbol{\omega}_1 = \frac{1}{\sqrt{3}}(1, 1, -1)$.

6. Thornton & Marion, Problem 11-31

(Note: For a planar lamina as described in Problem 11-31, the relationship between the principal momenta of inertia is called the *perpendicular-axis theorem.*)

• From Eq.(11.117) or (11.120) with $\mathbf{N} = 0$, $I_1 = I_2 \cos 2\alpha$ and $I_3 = I_2(1 + \cos 2\alpha)$, one gets $\dot{\boldsymbol{\omega}} = (\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3) = (-\omega_2\omega_3, \omega_3\omega_1, -\omega_1\omega_2 \tan^2\alpha)$. With the initial condition $\boldsymbol{\omega}(0) = (\Omega\cos\alpha, 0, \Omega\sin\alpha)$ and with some algebra, one can show $\dot{\omega}_2^2 = \omega_3^2\omega_1^2 = (-\omega_2^2 + \Omega^2\sin^2\alpha) \cdot \frac{-\omega_2^2 + \Omega^2\sin^2\alpha}{\cot^2\alpha} = (\omega_2^2 - \Omega^2\sin^2\alpha)^2 \tan^2\alpha$.

7. Thornton & Marion, Problem 11-33

(Note: For Problem 11-33, recall the video you watched in the class. You must reference your sources appropriately with a proper citation convention, and your answer must be your own work in your own words. Sources like Wikipedia or YouTube are *not* scientific literatures. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

• The video used in the class can be found at: http://www.youtube.com/watch?v=RtWbpyjJqrU

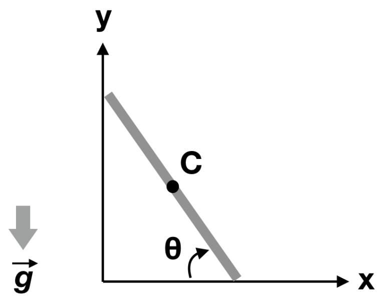
8. A ladder of mass M and length $2b$ — which can be considered as a slab of uniform density — is placed against a frictionless vertical wall (see the figure), with its bottom end on a frictionless horizontal floor. The ladder is released from the rest when the inclination angle θ is θ_0.

(a) Describe the subsequent motion of the ladder by obtaining its equation(s) of motion using energy conservation.

(b) Repeat (a) using the Lagrangian method with undetermined multipliers.

(c) Find the distance the ladder's top end has traveled when it ceases to touch the wall. What is the reaction (normal force) from the floor at that moment?

• (a) While the ladder's top end touches the wall, the center of mass C is at $(b\cos\theta, b\sin\theta)$. From this, $T = T_{\rm CM} + T_{\rm rot} = \frac{2}{3}Mb^2\dot{\theta}^2$ and $U = Mgb\sin\theta$. The generalized coordinate θ can uniquely specify the motion of the ladder until the ladder is detached from the wall, and the Lagrange's equation of motion gives $\frac{4}{3}b\ddot{\theta} = -g\cos\theta$. This can also be acquired by differentiating the energy conservation equation, $\frac{2}{3}Mb^2\dot{\theta}^2 = Mgb(\sin\theta_0 - \sin\theta)$.



• (b) Alternatively, the motion can be described with (x, y, θ) where x and y are C's coordinates. With $T = T_{\rm CM} + T_{\rm rot} = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{6}Mb^2\dot{\theta}^2$, $U = Mgy$ and the constraint equations $g_1(x, y, \theta) = x - b\cos\theta = 0$ and $g_2(x, y, \theta) = y - b\sin\theta = 0$, the Lagrange's equations of motion with undetermined multipliers become: *(i)* $-M\ddot{x} + \lambda_1 = 0$, *(ii)* $-Mg - M\ddot{y} + \lambda_2 = 0$, *(iii)* $-\frac{1}{3}Mb^2\ddot{\theta} + \lambda_1 b\sin\theta - \lambda_2 b\cos\theta = 0$. One can then acquire the same equations in (a). The physical meanings of λ_1 and λ_2 are the reactions from the wall and the floor, respectively.

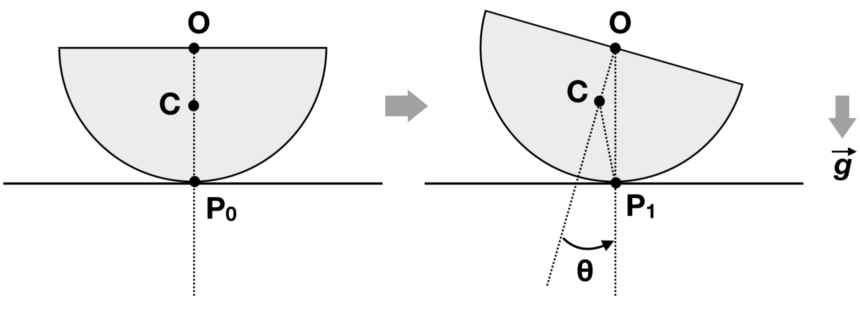
• (c) Combining the results from above, the reaction from the wall is $\lambda_1 = M\ddot{x} = -M(b\ddot{\theta}\sin\theta + b\dot{\theta}^2\cos\theta) = M\left(\frac{3}{4}g\sin\theta\cos\theta - \frac{3}{2}g\sin\theta_0\cos\theta + \frac{3}{2}g\sin\theta\cos\theta\right) = \frac{3}{4}Mg\cos\theta(3\sin\theta - 2\sin\theta_0)$. The ladder ceases to touch the wall when $\lambda_1 = 0$.

9. Consider a uniform hemisphere of radius R and total mass M, which lies on a horizontal plane in a uniform gravitational field g. The hemisphere is given a displacement in θ (see the figure) which starts small oscillations in the fixed plane of a great circle.

(a) Find the moment of inertia about the axis passing through the center of mass C.

(b) Determine the frequency of small oscillations if the hemisphere cannot slide on the plane. You will first need to find the location of C with respect to P_0 (a point fixed on the ground; see the figure) as a function of θ.

(c) Determine the frequency of small oscillations if the plane is frictionless (i.e., sliding can occur without friction).



• (a) With $\overline{OC} = \frac{3}{8}R$, $I_{{\rm about}\,O} = \frac{2}{5}MR^2 = I_{{\rm about}\,C} + M\left(\frac{3}{8}R\right)^2 \;\rightarrow\; I_{{\rm about}\,C} = \frac{83}{320}MR^2$

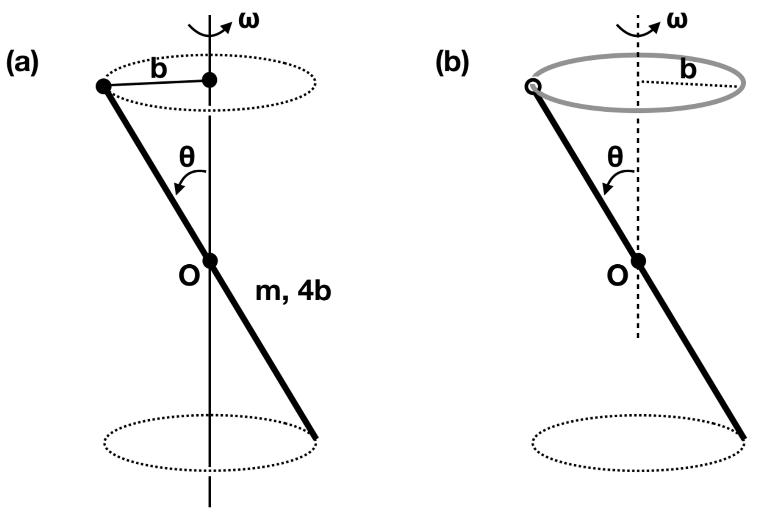
• (b) In the $x - y$ coordinate with origin P_0, C is located at $(R\theta - \overline{OC}\sin\theta, R - \overline{OC}\cos\theta) = R(\theta - \frac{3}{8}\sin\theta, 1 - \frac{3}{8}\cos\theta)$. From this, $T = T_{\rm CM} + T_{\rm rot} = \frac{1}{2}MR^2\left(\frac{73}{64} - \frac{3}{4}\cos\theta\right)\dot{\theta}^2 + \frac{83}{640}MR^2\dot{\theta}^2$ and $U = MgR(1 - \frac{3}{8}\cos\theta)$. And the Lagrange's equation of motion gives $MR^2\ddot{\theta}(\frac{7}{5} - \frac{3}{4}\cos\theta) = -\frac{3}{8}MgR\sin\theta(1 + \frac{R}{g}\dot{\theta}^2)$. Ignoring the higher order terms, one finds $\omega = \sqrt{\frac{15g}{26R}}$.

• (c) C is now located at $R(0, 1 - \frac{3}{8}\cos\theta)$, always right above P_0, as the gravity causes C to move only vertically. Thus, $T = T_{\rm CM} + T_{\rm rot} = \frac{9}{128}MR^2\dot{\theta}^2\sin^2\theta + \frac{83}{640}MR^2\dot{\theta}^2$ and $U = MgR(1 - \frac{3}{8}\cos\theta)$. And the Lagrange's equation of motion gives $\frac{83}{320}MR^2\ddot{\theta}(1 + \frac{45}{83}\sin^2\theta) = -\frac{3}{8}MgR\sin\theta(1 + \frac{3R}{8g}\dot{\theta}^2\cos\theta)$. Ignoring the higher order terms, one finds $\omega = \sqrt{\frac{120g}{83R}}$.

10. A thin uniform rod of mass m and length $4b$ spins with constant angular velocity ω about a vertical axis.

(a) First let us ignore gravity. The midpoint of the rod, O, act as a pivot point on the vertical axis allowing the rod to rotate without friction (but not slide along the rod). The rod makes a constant angle with the axis ($\theta = 30°$) and is kept in its motion by a string of length b that pulls the top end of the rod and is perpendicular to the axis (see figure (a)). Determine the tension in the string.

(b) Now consider gravity of constant acceleration g. The physical vertical axis no longer exists, but the rod still spins with constant ω about the vertical (see figure (b)). The top end of the rod is threaded on a frictionless horizontal ring of radius b (fixed in space) and slides freely in the ring. The rod moves in such a way that its center of mass is motionless on the vertical while its top end slides on the ring. As a result, the rod makes a constant angle with the vertical ($\theta = 30°$). Find the frequency ω allowed for this motion.



• (a-1) We choose a principal axes body coordinate system with its origin O, and with $\hat{\mathbf{e}}_3$ along the rod towards the top end, $\hat{\mathbf{e}}_2$ along $\mathbf{L}$, and $\hat{\mathbf{e}}_1$ according to the right-hand rule. Then, with $\theta = \frac{\pi}{6}$

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega} = \frac{4}{3}mb^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \omega \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{2}{3}mb^2\omega \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

• (a-2) From Eq.(11.117) or (11.120) with $\dot{\boldsymbol{\omega}} = I_3 = \omega_1 = 0$, the torque needed is $\mathbf{N} = N_1\hat{\mathbf{e}}_1 = -I_2\omega_2\omega_3\,\hat{\mathbf{e}}_1 = -\frac{1}{\sqrt{3}}mb^2\omega^2\,\hat{\mathbf{e}}_1 = \mathbf{r} \times \mathbf{T} = -2bT\sin\left(\frac{2\pi}{3}\right)\hat{\mathbf{e}}_1 = -\sqrt{3}bT\hat{\mathbf{e}}_1$. Thus, $T = \frac{1}{3}mb\omega^2$.

• (b) The reaction to the rod's weight by the ring is directed vertically upward. This reaction force provides the torque needed for the rotation. Therefore, $\mathbf{N} = N_1\hat{\mathbf{e}}_1 = -\frac{1}{\sqrt{3}}mb^2\omega^2\,\hat{\mathbf{e}}_1 =$

$\mathbf{r} \times (-m\mathbf{g}) = -2b\, mg \sin\left(\frac{\pi}{6}\right) \hat{\mathbf{e}}_1 = -mgb\, \hat{\mathbf{e}}_1$. Thus, $\omega = 3^{\frac{1}{4}}\sqrt{\frac{g}{b}}$.

11. Let us take another look at Problem 9-64 from Homework #1. This time consider the effect of the Coriolis force on the rocket's trajectory. Assuming that the rocket is launched at a latitude of 45°N, find the horizontal deflections of the rocket at its maximum height for the cases (a) to (d). Make — and justify — appropriate assumptions if desired.

• (a) Using the rotating frame in Figure 10-9, and ignoring the velocity components other than in $\hat{\mathbf{e}}_z$ direction, the horizontal deflection by the Coriolis force is dominated by the component in $(-)\hat{\mathbf{e}}_y$ direction. Then, $\ddot{y} = -2\omega\dot{z}\cos\lambda$.

• (b) As one moves the rocket in $\hat{\mathbf{e}}_z$ direction by solving the differential equation of motion $\ddot{z} = \frac{u\alpha}{m} - g - \frac{c_{\rm w}\rho A\dot{z}^2}{2m}$, one may simultaneously acquire the deflection in $\hat{\mathbf{e}}_y$ direction, $d_y(t)$, from $\ddot{y} = -2\omega\dot{z}\cos\lambda + \frac{c_{\rm w}\rho A\dot{y}^2}{2m}$.

12. In the class we discussed various properties of the inertia tensor.

(a) Starting from a transformation Eq.(11.45) between a center of mass body coordinate system and another body system with the same coordinate orientations, prove Eq.(11.49), a generalized form of the *parallel-axis theorem*.

(b) Starting from a coordinate transformation for vectors, Eq.(11.55), follow the texts in Section 11.7 and review the properties of the inertia tensor — which were briefly discussed in the class and left for your exercise. In particular, prove explicitly *(i)* that an inertia tensor transforms between different coordinates as Eq.(11.63) (*similarity transformation*), *(ii)* that, for any inertia tensor, it is possible to perform a rotation of the coordinate axes in such a way that the inertia tensor becomes diagonal, *(iii)* that, if the three principal moments are all different, then the direction of the three principal axes are uniquely determined and form an orthogonal set, *(iv)* that, if two of the principal moments are equal, then the corresponding two principal axes are not uniquely determined but can be chosen so that the three principal axes do form an orthogonal set, *(v)* that, if all three principal moments are equal, then any axis is a principal axis with the same principal moment, and *(vi)* that the principal moments and the principal axis vectors are real.

• For (b)-*(v)* see the discussion at the end of Example 11.6.