Classical Mechanics II (Fall 2020): Final Examination

Dec. 12, 2020

[total 25 pts, closed book/cellphone, no calculator, 90 minutes]

• By 21:00pm, you are asked to (1) sign the Honor Code, (2) work on the following questions, (3) scan your answers, and (4) turn in your answers to eTL as a single pdf file (eTL does not accept multiple files). You have to stay in the meeting room and be visible to the TAs for the whole 90 minutes even if you have nothing to write down. Have your cellphone and student ID nearby. Prepare to pick up the phone and follow the TA's instructions. TA may randomly call you and ask to e.g., show your ID to the camera. TA may also call you to answer your in-private questions.

• It is *strongly* advised that you start to work on scanning and uploading your answer sheets by no later than 20:50pm. If you submit your work after 21:00pm, your score will be reduced by 2 point for each minute that the submission is late. If, for some reason, you experience technical difficulties on eTL, email your answer as a single pdf file to the lecturer (mornkr@snu.ac.kr) immediately. The time the email is *received* will be marked as your submission time.

• First, bring out a piece of paper and make a cover page (first page) of your answer sheets. Copy the following in your own handwriting and sign. Submit this as part of your answer.

명예규율 (Honor Code) 서약서

교과목명: 역학 2
담당교수: 김지훈
시험일: 2020년 12월 12일

나 (   )은/는 위 교과의 시험시간 중 어떠한 종류의 부정행위도 하지 않았음을 서약합니다.

소속 (학부/과):
학번:
이름:
서명 (혹은 도장):

• Then, make sure to prepare *at least* 6 pieces of A4 papers for your answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from ① to ⑥ on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet ②). If you need additional papers, write down the number clearly so the grader can locate your answers without difficulty. After the exam, you will turn in all 6 answer sheets, even if some are still blank.

• Your original answer sheets should be on physical A4 papers. Keep your original answer sheets on papers for your record. You may be asked to submit the original any time. Do not make your answer sheets digitally using a tablet/iPad and an electronic pencil. Electronic calculators will not be needed.

• Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you have any issue or question on the problem itself or on English expressions, you *must* raise it to your microphone in the first 30 minutes.

• Make your writing easy to read, and double check your scanned answers before submitting it to eTL. Illegible answers will *not* be graded.

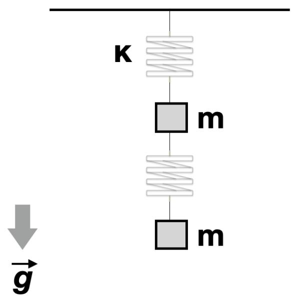
1. [4 pt] Consider a system of two identical masses of negligible sizes and mass m each which are connected to the ceiling by two massless springs of spring constant κ and original unstretched length l each, as shown in the figure below. The masses are constrained to move only vertically in the uniform gravitational field g.

(a) [1 pt] When both masses are at rest, find their equilibrium positions from the ceiling.

(b) [1 pt] Assuming small oscillations, find the system's Lagrangian and equations of motion. (Note: You may want to explain or justify why you do not have to consider the gravitational potential energies of the masses, when you adopt a coordinate for each mass measured from its equilibrium position.)

(c) [1 pt] Determine the eigenfrequencies and describe the corresponding normal mode motions.

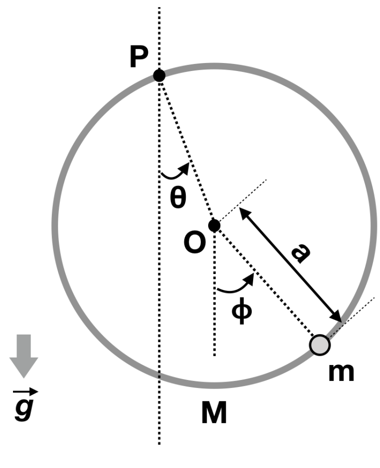
(d) [1 pt] Find an example initial condition for the system that later makes the two masses oscillate in the *symmetrical* normal mode.



2. [5 pt] A bead of mass m is threaded on a frictionless, circular wire hoop of mass M and radius a. The hoop is pivoted on its rim at point P (see the figure) so it can swing freely in its plane in the presence of the uniform gravitational field g.

(a) [1 pt] Determine the Lagrangian of the system using the generalized coordinates θ and ϕ shown in the figure.

(b) [1 pt] Now assuming small oscillations about the equilibrium points $\theta = \phi = 0$, find Lagrange's equations of motion of the system.



[Problem 2 continues in the next page.]

(c) [2 pt] Determine the eigenfrequencies and describe the corresponding normal mode motions.

(d) [1 pt] Finally let us carry out a quick sanity check on your answer in (c). When $m \ll M$, the system becomes a simpler physical pendulum of mass M. Assuming small oscillations about the equilibrium point $\theta = 0$, find $\theta(t)$ and its oscillation frequency. Verify that, in the limit $m \ll M$, the displacement $\theta(t)$ found in (c) reduces to what you acquired here in (d).

3. [5 pt] A string of length L is fixed at both ends, and is initially given no displacement but set into motion by being struck over a length $2s$ about its center. The center section is given an initial velocity v_0 at $t = 0$.

(a) [2 pt] Let us attempt to describe the subsequent motion of the string in terms of normal modes. For this, you may want to find the displacement $\Psi(x,t)$ as a function of position x and time t, and the characteristic frequencies ω_r (for the rth normal mode). For our trial solution $\Psi(x,t) = \sum_r \beta_r \psi_r(x) \chi_r(t)$, propose your choices for $\psi_r(x)$ and $\chi_r(t)$, and explain your reasoning.

(b) [2 pt] Acquire the exact form of $\Psi(x,t)$ and ω_r for the string described above.

(c) [1 pt] Among the "harmonics" of characteristic frequencies, which are missing when describing the vibration here? Briefly explain why.

4. [4 pt] The Lorentz transformation matrix between two inertial reference frames K and K' which move along their x_1- and x_1'-axes with a uniform relative velocity v, can be written as

$$\boldsymbol{\lambda} = \lambda_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix},$$

where $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ with indices μ, $\nu = 1, 2, 3, 4$. The quantity $\mathbb{X} = x_\mu = (\mathbf{x}, ict)$ is a four-vector as each of its components transforms according to the relation $\mathbb{X}' = x'_\mu = \boldsymbol{\lambda}\mathbb{X}$, with a 3-dimensional position vector $\mathbf{x}$ and time t.

(a) [1 pt] Prove that one can write a four-vector momentum of a particle (of mass m, 3-dimensional relativistic momentum $\mathbf{p}$, relativistic total energy E) as $\mathbb{P} = (\mathbf{p}, i\frac{E}{c})$. (Note: You will have to explicitly demonstrate that $\mathbb{P}$ is a four-vector. You may first want to show that $\mathbb{V} = \frac{d\mathbb{X}}{d\tau} = (\gamma\mathbf{u}, i\gamma c)$ is a four-vector that transforms according to the same $\boldsymbol{\lambda}$ above. Here τ is the Lorentz invariant proper time, and $\mathbf{u}$ is the 3-dimensional velocity vector of the particle.)

(b) [1 pt] By first showing $\mathbb{V}^2 = -c^2$, prove $\mathbb{P}^2 = -m^2c^2$. Show also that this leads to a familiar equation, $E^2 = p^2c^2 + m^2c^4$, where $p^2 = \mathbf{p}^2$.

[Problem 4 continues in the next page.]

(c) [2 pt] Show that the equation

$$\nabla^2 \Psi(\mathbf{x}, t) - \frac{1}{c^2}\frac{\partial^2 \Psi(\mathbf{x}, t)}{\partial t^2} = 0$$

is invariant under the Lorentz transformation. What is the physical meaning you can draw from this observation? (Note: You may want to simplify the math by demonstrating that the above equation is equal to

$$\sum_{\mu=1}^{4} \frac{\partial^2 \Psi(x_\mu)}{\partial x_\mu^2} = 0,$$

and then proceed to show $\sum_\mu \frac{\partial^2}{\partial x_\mu^2} = \sum_\mu \frac{\partial^2}{\partial x_\mu'^2}$.)

5. [4 pt] A particle (particle 1) of mass m_1 and relativistic total energy E_1 decays in flight into two identical particles (particle 2 and 3) of mass m_2. (An example would be a neutral pion π^0 decaying into two photons.) θ_2 and θ_3 are the LAB angles particle 2 and 3 emerges at, respectively, relative to the particle 1's direction. Consider two scenarios below.

(a) [2 pt] In the first scenario, the two created particles emerge on each side of the particle 1's direction with equal angles $\theta_2 = \theta_3 = \theta$. Find $\cos\theta$ as a function of m_1, m_2 and E_1.

(b) [2 pt] In the second scenario, particle 2 is emitted at $\theta_2 = 90°$. Find the energies of the created particles (E_2 and E_3 for particle 2 and 3, respectively) as functions of m_1, m_2 and E_1. (Note: You may expedite your calculation by using the conservation of four-vector momenta, $\mathbb{P}_1 = \mathbb{P}_2 + \mathbb{P}_3$, while noticing $\mathbb{P}^2 = -m^2c^2$ derived in Problem 4.(b) above.)

6. [3 pt] Throughout the semester we discussed many examples in which concepts in classical mechanics are utilized in explaining daily phenomena. We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques like order-of-magnitude estimation and/or dimensional analysis. For the two questions here, you have been given a fair amount of heads-up during the classes.

(a) [2 pt] In the last class of the semester, five of your peers presented their term projects. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 2-3 sentences is expected to clearly convey the core physics idea of his/her term project. If you were one of the presenters, please choose someone else's.

(b) [1 pt] Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem set-up. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to the fellow physics major student in ~3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.