

Load distribution in weighted complex networks

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We study the load distribution in weighted networks by measuring the effective number of optimal paths passing through a given vertex. The optimal path, along which the total cost is minimum, crucially depends on the cost distribution function $p_c(c)$. In the strong disorder limit, where $p_c(c) \sim c^{-1}$, the load distribution follows a power law both in the Erdős-Rényi (ER) random graphs and in the scale-free (SF) networks, and its characteristics are determined by the structure of the minimum spanning tree. The distribution of loads at vertices with a given vertex degree also follows the SF nature similar to the whole load distribution, implying that the global transport property is not correlated to the local structural information. Finally, we measure the effect of disorder by the correlation coefficient between vertex degree and load, finding that it is larger for ER networks than for SF networks.

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The study of complex systems in the framework of the network representation has attracted considerable attention as an interdisciplinary subject [1–4]. Of particular interest is the emerging pattern, a power-law behavior in the degree distribution $p_d(k) \sim k^{-\gamma}$, where the degree k is the number of edges connecting a given vertex. Such complex networks are called scale-free (SF) networks [5]. Transport phenomena on SF networks such as data packet transport on the communication network [6–11], random walks [12], and the information exchange in social networks [13] are of vital importance from both theoretical and practical perspectives. As the first step, the transport property can be studied through the quantity called the load introduced recently [14], or the betweenness centrality in social network literature [15]. To be specific, a packet leaves and arrives between a pair of vertices, traveling along the shortest pathway(s) between the pair. When the shortest pathways branch, the packet is assumed to be divided evenly. Then, the load ℓ_i of a vertex i is defined as the accumulated sum of the amount of packets passing through that vertex when every pair of vertices send and receive a unit packet. The load thus quantifies the level of burden of vertices in the shortest path-based transport processes. The load distribution of SF networks also follows a power law $p_L(\ell) \sim \ell^{-\delta}$, with the load exponent δ [14]. When packets travel with constant velocity, the numerics indicate that the effect of time delay on the load does not have effect on the shape of the load distribution. Therefore the load is usually measured as the effective number of pathways passing through a given vertex. So far, the power-law behavior of the load distribution was observed only on binary networks, where the strength of each edge is either 1 (present) or 0 (absent) [14].

To describe transport phenomena in a more realistic way, one has to take into account of the heterogeneity of elements, e.g., buffer sizes and/or bandwidths of each router or optical cable. For example, the Abilene network consists of high-bandwidth backbone, while their subconnecting systems do of low bandwidth [16]. In such weighted networks, the notion of the shortest path, the path with minimal number of

hops between two vertices in the binary network, may not be as appropriate as the so-called optimal path, the path over which the sum of costs becomes minimal. Thus it is natural to generalize the load to that based on the optimal paths in weighted networks. In general, the optimal paths in weighted networks often take a detour with respect to the shortest path to reduce the total cost. Such a detour makes the optimal path longer than the shortest path [17,18], which we expect leads to redistribution of the load, the characteristics of which depends on the cost distribution function. Here we will concentrate on the disorder in edges only, i.e., the case where edges carry their own nonuniform costs with a distribution $p_c(c)$. In a recent paper, Park *et al.* [19] studied a related problem based on the vertex cost, focusing on the relation between the cost and the load of a vertex, finding that the load of a vertex decreases exponentially with the vertex cost but scales as a power law with respect to the degree.

To study the effect of the weight in a systematic way, we consider a weighted network model by assigning random cost on each edge of the static model [14]. The static model network is composed of N vertices indexed by the integers $i = 1, 2, \dots, N$ and L edges that are added one by one avoiding the multiple connections: Each step, an edge between a vertex pair (i, j) is chosen with the probability $p_i p_j$ where $p_i = i^{-\alpha} / \sum_j j^{-\alpha}$ and added unless it already exists. α is a control parameter in the range $[0, 1)$ and we use $L = 2N$ in this work. The network thereby constructed is a SF network with the degree exponent $\gamma = 1 + 1/\alpha$. Note that the $\alpha = 0$ case corresponds to the random graph of Erdős and Rényi [20]. Next, cost of each edge is assigned randomly with a given distribution function, independent of the degrees of the vertices located at its ends.

Partly motivated by the recent observation on the real-world weighted networks [21,22], we consider a family of the cost distribution function in a power-law form $p_c(c) \sim c^{-\omega}$, where when $\omega > 1$ ($\omega < 1$), c is chosen as $c > 1$ ($0 < c < 1$). The limit $\omega \rightarrow \infty$ corresponds to the exponential cost distribution, and the limit $\omega \rightarrow 0$ does to the uniform cost

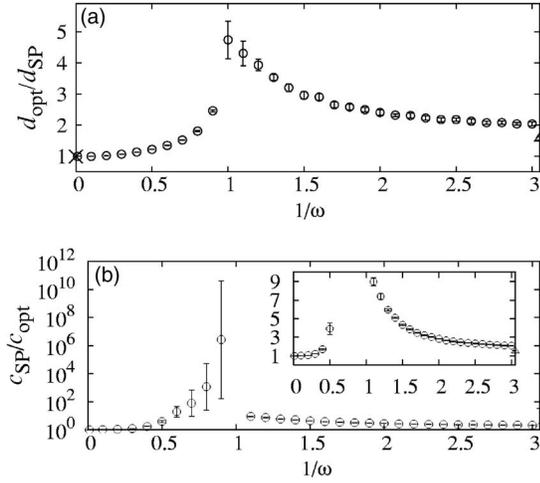


FIG. 1. The relative optimal path length $d_{\text{opt}}/d_{\text{SP}}$ (a) and the relative advantage $c_{\text{SP}}/c_{\text{opt}}$ (b) as a function of the exponent ω of the cost distribution for SF network with $\gamma=3$ and $N=10^4$. The effect of the disorder becomes maximal at $\omega=1$, which manifests itself as the peak in both quantities. The cross (triangle) symbol represents the the shortest path length of the underlying binary network (the uniform cost distribution case). The inset of (b) shows the same data in linear scale, excluding those with $c_{\text{SP}}/c_{\text{opt}} \geq 10^1$.

distribution. As ω is lowered from ∞ , the heterogeneity of the cost distribution increases and we expect that the effect of disorder would increase, that is, longer optimal path length d_{opt} . The disorder effect would be maximal at $\omega=1$, which corresponds to the so-called strong disorder limit [18,23], where the total cost is dominated by the maximum cost over the path. Note that if the costs are assigned as $\{c_i | c_1 < c_2 < \dots < c_L\}$ such that $\sum_{j=1}^n c_j < c_{n+1}$, for instance, $c_n = 2^{n-1}$, then one would get the strong disorder limit and in this case we have $\omega=1$. As ω is lowered further from $\omega=1$, the effect of disorder would decrease again. This expectation is confirmed by numerical simulations as shown in Fig. 1(a) specifically for the SF network with $\gamma=3$, where the ratio of the average optimal path length and the average shortest path length d_{SP} are shown. For $\omega \neq 1$, the network is in the weak disorder regime in the terminology of Ref. [18]. It is worthwhile to note that the disorder effect does not vanish qualitatively even for the uniform distribution, the $\omega \rightarrow 0$ limit. It is not obvious if the crossover from $\omega=1^+$ to $\omega=1^-$ is continuous or discontinuous under the present data, the test of which would require much larger system sizes.

As the optimal path length increases, the total load of the system grows, as it satisfies the sum rule $\sum_i \ell_i = N(N-1)(d_{\text{opt}}+1)$. That is, by taking the optimal paths, we need more resource—router capacity, for example—to maintain the system in the free-flow state, where packets can travel without congestion. The advantage in taking the optimal paths then should compensate the increase of resource requirement, for it to be *optimal*. We show in Fig. 1(b) the advantage in taking the optimal paths, by the ratio of c_{SP} , the average cost along the shortest paths, to the c_{opt} , that for the optimal paths. As anticipated, the advantage is always larger than 1, and it is more advantageous to take the optimal paths as the disorder becomes strong, exhibiting a strong peak near

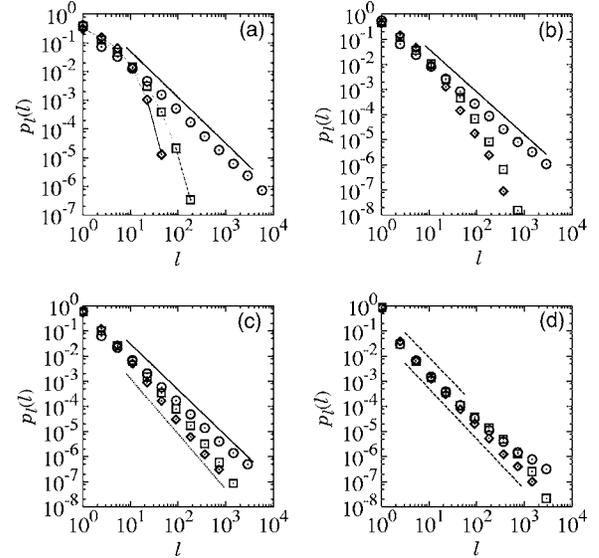


FIG. 2. The load distribution of the weighted networks in the presence of disorder. Plotted are in each panel for the ER networks (a), the SF networks with $\gamma=4$ (b), $\gamma=3$ (c), and $\gamma=2.0$ (d). All networks are of size $N=10^4$. Data are logarithmically binned. The symbols are for the exponential cost distribution (\diamond), the uniform cost distribution (\square), and the strong disorder limit (\circ). The guide-lines are plotted together, with the slope -1.6 (a), -1.7 (b), -1.8 (solid) and -2.2 (dotted) (c), and -2.0 (d).

$\omega=1$. Furthermore, $c_{\text{SP}}/c_{\text{opt}} > d_{\text{opt}}/d_{\text{SP}}$ in all cases studied, that is, the increase in the optimal path length (the required resource) is compensated by the cost advantage.

In the following, we focus on the three specific cases of the cost distribution; (i) the exponential cost distribution ($\omega \rightarrow \infty$), (ii) the uniform cost distribution ($\omega \rightarrow 0$), and (iii) the strong disorder limit ($\omega=1$).

Load distribution. First, in the most global level, we study the load distribution of the networks in the presence of the disorder. One may expect that the change in the load distribution would be largest for the strong disorder, moderate but substantial for the uniform cost distribution, and minimal for the exponential distribution, as was the case for the change in the optimal path length. This picture is confirmed by numerical simulations, shown in Fig. 2, for the ER network, the SF networks with $\gamma=4, 3$, and 2. The load of the weighted networks can be efficiently computed by the modified Dijkstra algorithm [24,25]. For the strong disorder limit ($\omega=1$), the equivalent calculation can be done by noting that the optimal paths in this case lie on the minimum spanning tree (MST) [18], constructed by removing links in the descending order of their costs one by one unless such a removal disconnects the graph. We exploit this fact and compute the load in the strong disorder limit by constructing MST via, e.g., the Kruskal algorithm [24]. There one can see that the load distribution becomes broader as the strength of the disorder increases.

A surprising result is that, in the strong disorder limit, the load distribution follows a power law even for the ER network [Fig 2(a)]. Recall that the ER network has the exponentially decaying load distribution in its binary version. The

load exponent is measured to be $\delta \approx 1.59(4)$ for the ER network. For the SF networks, it is measured that $\delta = 1.64(4)$, 1, 69(2), 1.73(3), 1.82(8), 1.95(5), and 1.96(5) for $\gamma = 5.0, 4.0, 3.5, 3.0, 2.5$, and 2.0, respectively. In the strong disorder limit, the MST can be regarded as the composition of the percolation clusters at the percolation threshold and interlinks, so called hot bonds, connecting disconnected clusters [18]. This picture can be applied for the case of $\gamma > 3$, where the percolation threshold is finite. The degree of each vertex in the MST is proportional to that in the original network [26]. For example, the degree distribution of the MST of the ER network is not scale-free. In this case, it is measured that the load exponent depends on the degree exponent. When $2 < \gamma < 3$, however, due to the vanishing percolation threshold [27] and the presence of non-negligible fraction of short cycles [28], the percolation cluster network picture no longer holds. The formation of the MST is not random and the proportionality of the degrees of the vertex in the MST and in the original network breaks down. In this case we find numerically that the load distribution follows a power law with the exponent about 2 with an additional fatter tail.

In the weak disorder limit, the sum of all the costs along the path determines the optimal path. The load distribution depends on the strength of disorder. Note that in this case, the load distribution for the ER network does not follow a power law. Numerical data for other values of γ are also shown in Fig. 2.

Load-degree scaling. Next, to probe loads in the more microscopic settings, we focus on how the load of an individual vertex would change by the presence of disorder. In the binary networks, there is a scaling relation between the degree and the load of a vertex, as

$$\ell_b \sim k^\eta, \quad (1)$$

with $\eta = (\gamma - 1) / (\delta - 1)$ [14], where the subscript b denotes the binary network. When the disorder becomes sufficiently strong, however, it is not clear if such a scaling relation would still hold. We find indeed there exists strong dispersion in the scaling relation for the uniform cost distribution. To characterize the dispersion, we consider the conditional probability $p(\ell_w | k)$ that the load of a vertex with degree k is ℓ_w . If this distribution is sufficiently broad, it is meaningless to speak of a scaling relation as in the binary network version. We show the result of numerical simulation specifically for $k=2$ in Fig. 3. In the strong disorder limit $p(\ell_w | k)$ even follows the similar power-law decay as $p_\ell(\ell_w)$, meaning that the degree of a vertex has essentially no correlations with the load. This picture also applies to the SF network, as shown in Fig. 3(b) for the case of $\gamma=3$. We also find that it is also the case for intermediate degree nodes, $k=10$, for example, in the SF networks with $\gamma=3$. Thus in the presence of disorder, one may not predict the level of traffic at a router or the centrality of an individual based solely on the connectivity information.

Disorder vs network heterogeneity. We now turn our attention to the interplay between the disorder in edge costs and the network heterogeneity. To characterize the extent of

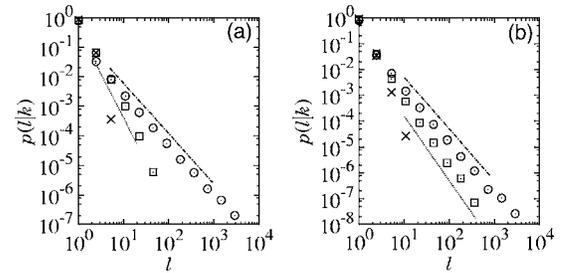


FIG. 3. The condition probability $p(\ell|k)$ with $k=2$ for the ER network (a) and for the SF network with $\gamma=3$ (b). Data are for the exponential cost distribution (\times), the uniform cost distribution (\square), and the strong disorder limit (\circ). Plotted are the guidelines with slopes -3 and -1.7 in (a); -2.5 and -2 in (b).

the effect due to the disorder by a simple scalar measure, we introduce the Pearson correlation coefficients between the degree (k) and the load (ℓ_w) in the weighted network, and between ℓ_w and the load (ℓ_b) in the binary network, of the same vertex. The correlation coefficient is defined by $r_{xy} \equiv (\overline{xy} - \bar{x}\bar{y}) / \sigma_x \sigma_y$, where \bar{x} and σ_x denote the average and the standard deviation, respectively, of a variable x over all vertices. (x, y) stands for (k, ℓ_w) or (ℓ_w, ℓ_b) .

As the scaling relation Eq. (1) implies, the correlation $r_{k\ell}$ for binary network is high, typically larger than 0.9. This correlation decreases as the strength of disorder increases and the effect of disorder is larger for the ER network than for the SF network.

The result that the effect of disorder is larger for the ER network than for the SF network may be understood as follows: The binary ER network is homogeneous, so are the pathways inside it. As we turn on the disorder, this homogeneity breaks down and the extent to which this induced heterogeneity becomes larger as the strength of disorder increases. Due to this induced heterogeneity, *de novo* hierarchy in vertices [29] builds up and according to this hierarchy, the pathway structure of the weighted ER network are rearranged. In SF networks, however, such heterogeneity and hierarchy in vertices exist even in the absence of disorder. Furthermore, such an inherent heterogeneity suppresses the effect of disorder in a way that the larger the degree of a vertex is, the more likely it is to take an edge with very small cost. As a result, the disorder in edge cost competes with the heterogeneity of the network to give a full effect, and thus the effect of disorder becomes weaker as the heterogeneity of a network increases (γ decreases), as can be seen in Fig. 4.

To summarize, we have studied the optimal transport in weighted complex networks by extending the notion of the load used in binary networks to weighted networks and investigated how the transport property is changed accordingly. We found that the load distribution in the strong disorder limit is related to the structure of the minimum spanning tree. The load distribution follows a power law even for the ER network, and the load exponent δ depends on γ for the SF network with $\gamma > 3$. For $2 < \gamma < 3$, however, $\delta \approx 2$ but with additional fatter tail. In the weak disorder regime, it is found that for sufficiently narrow cost distribution the load

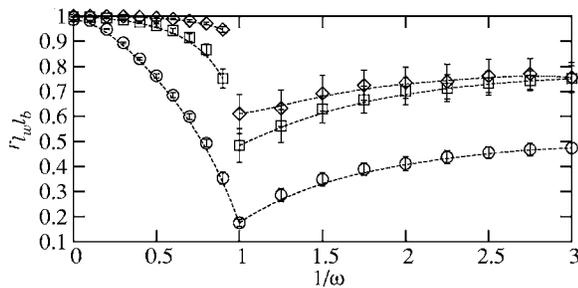


FIG. 4. The correlation coefficient $r_{\ell_w \ell_b}$ as a function of ω for the ER network (\circ), the SF network with $\gamma=3$ (\square), and $\gamma=2$ (\diamond). The dotted lines interpolating the data are drawn for the eye. The plot of $r_{k \ell_w}$, which is not shown here, follows very similar feature, owing to the high correlation between k and ℓ_b .

distribution changes little. As the heterogeneity of weights increases, however, the effect becomes significant and becomes maximal as it approaches the cost distribution $p_c(c) \sim 1/c$, being equivalent to the strong disorder limit. This

situation also holds on the individual vertex level, in that the fluctuation of the load of the vertices of a given degree grows unboundedly as the cost distribution becomes broader. The effect of disorder is manifestly larger for the ER networks than in the SF networks, since the disorder must compete with the network heterogeneity in SF networks. Finally, we note that the time delay effect by packets travelling with constant speed does not change the load exponent even in weighted networks, which is checked through the weighted ER network.

In this Brief Report, we have considered the uncorrelated cost distributions on uncorrelated networks only. In the real world as well as the model evolving networks, however, the weight of a link and the degrees of the vertices at each end are often correlated [21]. The effect of such correlated disorder on the transport property in the weighted networks is an open question.

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