Flipped $SU(5)$ from $Z_{12-1}$ orbifold with Wilson line

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Received 8 September 2006; received in revised form 27 December 2006; accepted 1 February 2007

Abstract

We construct a three family flipped $SU(5)$ model from the heterotic string theory compactified on the $Z_{12-1}$ orbifold with one Wilson line. The gauge group is $SU(5) \times U(1)_X \times U(1)^3 \times (SU(2) \times SO(10) \times U(1)^2)'$. This model does not derive any non-Abelian group except $SU(5)$ from $E_8$, which is possible only for two cases in case of one shift $V$, one in $Z_{12-1}$ and the other in $Z_{12-II}$. We present all possible Yukawa couplings. We place the third quark family in the twisted sectors and two light quark families in the untwisted sector. From the Yukawa couplings, the model provides the $R$-parity, the doublet–triplet splitting, and one pair of Higgs doublets. It is also shown that quark and lepton mixings are possible. So far we have not encountered a serious phenomenological problem. There exist vector-like flavor $SU(5)$ exotics (including $Q_{em} = \pm \frac{1}{6}$ color exotics and $Q_{em} = \pm \frac{1}{2}$ electromagnetic exotics) and $SU(5)$ vector-like singlet exotics with $Q_{em} = \pm \frac{1}{2}$ which can be removed near the GUT scale. In this model, $\sin^2 \theta_W^0 = \frac{3}{8}$ at the full unification scale.

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PACS: 11.25.Mj; 12.10.Kt; 12.60.Jv

Keywords: Orbifold; Three families; Yukawa couplings; Flipped $SU(5)$

1. Introduction

At present, it is of utmost importance to connect the high energy string theory with the low energy standard model, in particular with the minimal supersymmetric standard model (MSSM).

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doi:10.1016/j.nuclphysb.2007.02.008
The initial attempt of the Calabi–Yau space compactification, which is geometrical, has been very attractive [1]. But the orbifold compactification, also being a geometrical device, got more interest due to its simplicity in model buildings [2,3]. Initially, the standard-like models were looked for [4], in an attempt to obtain minimal supersymmetric standard models (MSSMs), but it became clear that the standard-like models have a serious problem on \( \sin^2 \theta_W \) to arrive at MSSMs [5]. All \( \mathbb{Z}_N \) models without Wilson lines were tabulated a long time ago [6] and recently all \( \mathbb{Z}_3 \) models with Wilson lines are tabulated in a book [7].

The \( \sin^2 \theta_W \) problem is that it is better for the bare value of \( \sin^2 \theta_W^0 \) at the unification or string scale to be close to \( \frac{3}{8} \) [5] so that it reproduces the fact of the convergence of three gauge couplings at one point near the unification scale [8]. The so-called flipped SU(5) does not fulfill this requirement automatically due to the leakage of \( U(1)_Y \) beyond SU(5).\(^1\) Thus, the \( \sin^2 \theta_W \) problem directs toward grand unified theories (GUTs) from superstring without the electroweak hypercharge \( Y \) leaking outside the GUT group. In this regard, one may consider simple GUT groups SU(5) [12], SO(10) [13], \( E_6 \) [14] and trinification SU(3)\(^3\) [15]. The simplest orbifold without matter representations beyond the fundamentals require the Kac–Moody level \( K = 1 \). With \( K = 1 \), one cannot obtain adjoint representations [7]; thus among the above GUT groups the trinification group is the allowed one. Also, the Pati–Salam SU(4) \( \times \) SU(2) \( \times \) SU(2) [16] can be broken to the standard model without an adjoint matter representation; above the GUT scale however three gauge couplings of the Pati–Salam model diverge rather than evolving in unison. Thus, trinification GUT seems to be the most attractive solution regarding the \( \sin^2 \theta_W \) problem. The trinification is possible only in \( \mathbb{Z}_3 \) orbifolds [17].

Another interesting GUT group, though not giving \( \sin^2 \theta_W = \frac{3}{8} \) necessarily, is the flipped SU(5) [9,10] where the exchanges \( d^c \leftrightarrow u^c \) and \( e^c \leftrightarrow (\text{neutral singlet } \nu^c) \) in the representations of SU(5) are adopted. The matter representation of the flipped SU(5) is, under SU(5) \( \times \) U(1)\(_X\),\(^2\)

\[
16_{\text{flip}} \equiv 10_1 + \bar{5}_{-3} + 1_5.
\] (1)

The electroweak hypercharge is given by

\[
Y = \frac{1}{5} (X + Y_5)
\] (2)

where \( Y_5 = \text{diag}(\frac{1}{3} \frac{1}{3} \frac{1}{3} -\frac{1}{2} -\frac{1}{2}) \) and \( X = \text{diag}(x \ x \ x \ x) \). Then, the electroweak hypercharges of 15 and \( \bar{5}_{-3} \) are +1, \(-\frac{2}{3}\), and \(-\frac{1}{2}\), which are \( e^c, u^c \), and electron doublet. There are some nice features of flipped SU(5) [18].

From the string context, flipped SU(5) was considered before in the fermionic construction scheme [11] and recently in orbifold construction also [19], Calabi–Yau compactification [20], and intersecting D-brane models [21]. Let us call flipped SU(5) from string construction ‘string flipped’ SU(5). In string flipped SU(5), it does not necessarily predict \( \sin^2 \theta_W = \frac{3}{8} \) at the unification scale [5]. However, if we introduce more parameters intrinsic in flipped SU(5), we may fit parameters so that the gauge couplings meet at one point at the string scale \( M_s \), the unification scale of SU(5) and U(1)\(_X\) couplings. These parameters include the symmetry breaking scale \( M_{\text{GUT}} \) for SU(5) \( \times \) U(1)\(_X\) \( \rightarrow \) SM breaking and intermediate scales of vector-like representations. Above \( M_{\text{GUT}} \) the RG evolutions of SU(5) and U(1)\(_X\) couplings are different, and we do not expect a string scale around \( 0.7 \times 10^{18} \) GeV [22] but can be determined by the unification

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\(^1\) The terminology flipped SU(5) was used as an SU(5) \( \times \) U(1)\(_X\) subgroup of SO(10) [9–11]. In this paper, we still use the same terminology if there appear 16s having the same quantum numbers as in the flipped SU(5).

\(^2\) The quantum number \( X \) of U(1)\(_X\) in the flipped SU(5) is highlighted.
ansatz at $M_s$ and mass scales of vector-like representations [23]. In our string orbifold model, $\sin^2\theta_W$ turns out to be $\frac{3}{8}$ at the full unification scale due to the possibility of the electroweak hypercharge embedding in $SO(10)$. Thus such vector-like fields should be removed near the GUT scale.

String models in general include exotics. Electromagnetic exotics (E-exotics) are fractionally charged particles which are non-Abelian gauge group singlets. Color exotics (C-exotics) are quarks with non-standard charges, i.e. color triplet quarks not having $Q_{em} = 2/3$ or $-1/3$ and color anti-triplet quarks not having $Q_{em} = -2/3$ or $1/3$. Flipped $SU(5)$ GUT exotics (G-exotics) are $SU(5)$ representations, not having the $X$ charges of $10_1, \bar{10}_{-1}, 5_3, \bar{5}_{-3}, 5_{-2}, \bar{5}_2, 1_{\pm 5}, 1_0$. G-exotics contain C-exotics and fractionally charged leptons.

The fermionic construction of flipped $SU(5)$ has shown the existence of E-exotics with $Q_{em} = \pm \frac{1}{2}$ and integer charged ‘cryptons’ where cryptons are defined to be the composites of the hidden sector confining group $SU(4)'$ [24]. Cosmological effect of cryptons was given in [25]. Because of the possibility that fractionally charged particles exist in most string vacua, discovery of fractionally charged particles may strongly hint the correctness of the idea of string compactification.

In this paper, we present the orbifold compactification with $Z_{12}^{-I}$ twist. This contains a detailed account of Ref.[19]. In addition, we present another orbifold model having a hidden sector $SU(4)'$. We succeeded in constructing a phenomenologically desirable flipped $SU(5)$ model from $Z_{12}^{-I}$.

We need three families of $16_{\text{flip}}$, where

$$16_{\text{flip}} \equiv 10_1 + \bar{5}_{-3} + 1_5 = (d^c, q, v^c) + (u^c, l) + e^c.$$  \hspace{1cm} (3)

For spontaneous symmetry breaking, we need also the Higgs fields,

$$(10_1 + \bar{10}_{-1}) + (5_{-2} + \bar{5}_2).$$ \hspace{1cm} (4)

Sometimes, it is useful to represent the components in terms of

$$10_1: \left( \begin{array}{c} d^c \\ q \\ v^c \end{array} \right), \quad \bar{5}_{-3}: \left( \begin{array}{c} u^c \\ l \end{array} \right), \quad 1_5: \ e^c,$$

$$5_{-2}: \left( \begin{array}{c} D \\ h_d \end{array} \right), \quad \bar{5}_2: \left( \begin{array}{c} \bar{D} \\ h_u \end{array} \right).$$ \hspace{1cm} (5)

where $q$ and $l$ are lepton and quark doublets, $D$ is $Q_{em} = -\frac{1}{3}$ quark, and $h_{d,u}$ are Higgs doublets giving mass to $d, u$ quarks. Spontaneous symmetry breaking of flipped $SU(5)$ proceeds via VEVs of $10_1$ and $\bar{10}_{-1}$ (components $\langle v^c \rangle, \langle \bar{v}^c \rangle$) and $5_{-2}$ and $\bar{5}_2$ (components in $\langle h_d \rangle, \langle h_u \rangle$).

In this model, there appear two light families from the untwisted sector and the third heavy family from twisted sectors. This is dictated from the Yukawa coupling structure. It also leads to

(i) the doublet–triplet splitting,
(ii) one pair of Higgs doublets, and
(iii) the existence of $R$-parity.

Some standard models from fermionic construction are worthwhile to note since they may be free from some problems of GUTs [26]. But these models use $Z_2 \times Z_2$ which may have a very different phenomenology from the one we discuss here with $Z_{12}^{-I}$.

In Section 2, we present a review on orbifold construction with order $N = 12$. Here we include formulae with Wilson lines also. In Sections 3 and 4, the untwisted and twisted sector spectra are calculated in detail. In Section 5, we collect all observable sector fields. In Section 6, we present the Yukawa coupling structure and derive some phenomenological consequences. In Section 7 we show $\sin^2\theta_W^0 = \frac{3}{8}$. Section 8 is a conclusion. In Appendix A, we provide another model having $SU(4)'$.  


2. Orbifold method

An orbifold is constructed from a manifold by identifying points under a discrete symmetry group. The six internal space is orbifolded by a twist vector $\phi_s$. With three complexified components, $\phi_s$ has three components $\phi_{s1}$, $\phi_{s2}$ and $\phi_{s3}$. The twist of $\mathbb{Z}_{12-1}$ orbifold is \[ \phi_s = \left( \frac{5}{12}, \frac{4}{12}, \frac{1}{12} \right) \] with $\phi_s^2 = \frac{1}{12} \cdot \frac{7}{2}$.

Torus corresponding to $\phi_{si}$ is twisted by $\phi_{si}$. Hence the first and third tori have one fixed point while the second torus being modded by $\mathbb{Z}_3$ has three fixed points as schematically shown in Fig. 1. Multiplicity due to fixed points are three. These can be distinguished by Wilson lines.

In ten-dimensional (10D) heterotic string, left and right movers are treated as gauge group degrees and $\mathcal{N} = 1$ supersymmetry, respectively. The embedding possibility is in the gauge group space of left movers, NS sector of left and right movers, and R sector of right movers. So we focus on the embedding in the group space for left movers and in the R sector for right movers. In 10D, the R sector embedding is given by $\phi_s$. The group space embedding is given by sixteen numbers, $V^I (I = 1, 2, \ldots, 16) \equiv \{ v(I = 1, \ldots, 8), v'(I = 9, \ldots, 16) \}$. Factoring out $\frac{1}{12}$ by defining $\phi_{sa} = \frac{1}{12} \phi_a$, $v_a = \frac{1}{12} w_a$, $v'_a = \frac{1}{12} w'_a$, we must satisfy for a $\mathbb{Z}_{12-1}$ orbifold \[ \sum_{a=1}^{3} \phi_{a}^2 - \sum_{a=1}^{8} w_{a}^2 - \sum_{a=1}^{8} w'_{a}^2 = 0 \mod 24, \]

\[ 3a_3 = 3a_4 = 0, \quad a_1 = a_2 = a_5 = a_6 = 0. \] (7)

2.1. Dynkin diagram technique for finding gauge group

Just for finding out a gauge group structure, the Dynkin diagram technique is extremely useful [28]. In the Dynkin diagram, each simple root is endowed with a Coxeter label. A Dynkin diagram technique of obtaining maximal subgroups is to strike out a simple root from the extended Dynkin diagram. In orbifold, this is generalized to strike out some roots where sum of the eliminated Coxeter labels add up to order $N$ of $\mathbb{Z}_N$. To have $SU(5)$ only without a Wilson line, there must remain four linearly connected simple roots. So, for $N \leq 8$ orbifold, it is impossible.\footnote{By a two step process using Wilson lines, it is possible to obtain $SU(5)$ in other $\mathbb{Z}_N$ orbifolds.}

For $\mathbb{Z}_{12}$, there is only possibility. Suppose $\sum_i c_i = N$ where $c_i$ is the Coxeter label of simple root $\alpha_i$. There is only one possibility which is $c_0 + c_1 + c_2 + c_6 + c_7 = 12$. See Fig. 2. Thus, from orbifold compactification, there are only two possibilities for constructing flipped $SU(5)$
models, one in $\mathbb{Z}_{12-1}$ and another in $\mathbb{Z}_{12-II}$. For $N = \sum_i c_i = N$, the shift vector $V$ is given by $V = \sum_i A_i$ where $A_i$ are fundamental weights [27]. Thus, for $\mathbb{Z}_{12}$ orbifolds $V$ is given by $V_1 = A_0 + A_1 + A_2 + A_6 + A_7$ with $A_0 = 0$. Thus, the shift vector for flipped $SU(5)$ is

$$V_1 = \frac{1}{12} \left( \frac{17}{2} \frac{5}{2} \frac{3}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} - \frac{1}{2} \right) (\cdots).$$

Now we shift the origin by $-\frac{5}{24}$, to obtain

$$V_2 = \frac{1}{12} \left( 11 \frac{5}{4} 3 \frac{3}{3} \frac{3}{2} \cdots \right). \quad (8)$$

Both $V_1$ and $V_2$ give an unbroken $SU(5)$. But the entries of $V_1$ and $V_2$ do not have five common entries. So, we try to add an integer times $\mathbb{Z}_{12-1}$ shift $\phi_s$ so that the resulting entries manifestly show five common entries. In this way, of course the $SU(5)$ non-Abelian group is kept. Usually, if one adds $\phi_s$ to three entries of $E_8$, some non-Abelian groups are broken. So our strategy is to add a multiple of $\phi_s$ such that an $SU(5)$ survives. For this, we add $(\frac{4}{12}, 0, \frac{8}{12}, 0, 0, 0, 0, \frac{4}{12})$ to obtain $\frac{1}{12} (155123336)$. Subtracting torus lattice and rearranging entries, we obtain

$$V_3 = \frac{1}{12} \left( 3 \frac{3}{3} \frac{3}{3} \frac{3}{5} 6 \frac{0}{0} \right) \quad \text{(9)}$$

which has five common entries. This form is perfectly simple enough in obtaining $SU(5)$ weights since there are five common entries. Otherwise, i.e. with $V_1$ or $V_2$, it is cumbersome to work out all the $SU(5)$ weights as tried out in [7].

Since the five entries are common, the simple roots for $SU(5)$ take the following standard form,

$$\begin{align*}
\alpha_1 &= (1 -1 \ 0 \ 0 ; \ 0 \ 0) , \\
\alpha_2 &= (0 \ 1 -1 \ 0 ; \ 0 \ 0) , \\
\alpha_3 &= (0 \ 0 \ 0 \ 0 \ 0) , \\
\alpha_4 &= (0 \ 0 \ 0 \ 0 \ 0) .
\end{align*} \quad (10)$$

Then, the highest weights of some representations we use are

5: \[
\begin{cases}
\left( 1 \ 0 \ 0 \ 0 \ 0 ; \ 0 \ 0 \right) , \\
\left( +\frac{1}{2} -\frac{1}{2} -\frac{1}{2} -\frac{1}{2} -\frac{1}{2} ; \ 0 \ 0 \right) ,
\end{cases} \quad (11)
\]

8: \[
\begin{cases}
\left( -1 \ 0 \ 0 \ 0 ; \ 0 \ 0 \right) , \\
\left( -\frac{1}{2} +\frac{1}{2} +\frac{1}{2} +\frac{1}{2} +\frac{1}{2} ; \ 0 \ 0 \right) ,
\end{cases} \quad (12)
\]

10: \[
\begin{cases}
\left( 1 \ 0 \ 0 \ 0 \ 0 ; \ 0 \ 0 \right) , \\
\left( +\frac{1}{2} +\frac{1}{2} -\frac{1}{2} -\frac{1}{2} -\frac{1}{2} ; \ 0 \ 0 \right) ,
\end{cases} \quad (13)
\]
Embedding of the orbifold action can be found by satisfying the modular invariance conditions (7). If one tries to have a specific gauge group, the Dynkin diagram technique is very helpful as we have discussed in Section 2.1. Or one can study the components of \((\cdots)(\cdots)\)' to guess the gauge group, but this method of finding gauge group is completed only after one obtains all nonzero roots of the gauge multiplet. For example, if one tries \(V_2\) of (8) and \(V_3\) of (9) then in both cases he will obtain \(SU(5)\). But guessing \(SU(5)\) from (8) is not straightforward. In this sense, the Dynkin diagram technique is superior. On the other hand, a computer search of gauge groups will cover all these cases. In the computer search, the identical shift vector is given by several different forms as done in \(V_2\) and \(V_3\). Usually, it is very difficult to identify all the same shift vectors [7].

Choosing the shift vector \((V)\) and Wilson lines \((a_1, \ldots, a_6)\) fixes the embedding of the orbifold action in the group space. So gauge groups and representations are fixed by the shift vector and Wilson lines, consistently with the modular invariance condition (7).

In summary, the compactification is specified by twisting represented by a shift form in the six internal space \(s\), shift \(V\), and Wilson lines \(a_1, \ldots, a_6\),

\[
\text{internal space: } \phi_s = (\phi_{s1}, \phi_{s2}, \phi_{s3}), \quad \text{group space: } V, \ a_i \ (i = 1, 2, \ldots, 6).
\]

Here, \(s_0\) determines the chirality and \(\tilde{s}\) encodes the orbifolding information of the R sector.

### 2.3. Massless modes

Finding out all the massless modes below the compactification scale is the key problem in the compactification process. The left movers and right movers have different relations for the Einstein mass-shell condition, even though the form has a similarity. In the symmetric orbifold [2], let us bosonize the Ramond sector of right movers, which is represented by four half integers in \(s\). The orbifold action is modding out the 6D torus, and \(s\) contains the orbifolding information of right movers under translation in the internal space. For left movers, momenta \(P\) corresponding to translation in the group space have the orbifold information. All these satisfy the level matching condition, \(M_L^2 = M_R^2\). Thus, left moving and right moving states on torus have the following vanishing vacuum energy for massless states,

\[
\text{left movers: } \frac{(P + kV)^2}{2} + \sum_j N_j^L \tilde{\phi}_j - \tilde{c}_k = 0, \\
\text{right movers: } \frac{(s + k\phi_s)^2}{2} + \sum_j N_j^R \tilde{\phi}_j - c_k = 0, \quad (16)
\]

where \(j\) runs over \(\{1, 2, 3, 1, 2, 3\}\). Here \(\tilde{\phi}_j \equiv k\phi_j \mod Z\) such that \(0 < \tilde{\phi}_j < 1\), \(\tilde{\phi}_j \equiv -k\phi_j \mod Z\) such that \(0 < \tilde{\phi}_j < 1\). (If \(k\phi_j\) is an integer, \(\tilde{\phi}_j = 1 [29,30]\).) For \(k = 0\), conditions for massless
left and right movers are given by

\[ P^2 = 2 - 2 \sum_j N^L_j \tilde{\phi}_j , \]

\[ s^2 = 1 - 2 \sum_j N^R_j \tilde{\phi}_j , \]

where \( \tilde{c}_0 = c_0 + \frac{1}{2} \).

The massless modes include graviton \( g_{\mu\nu} \), antisymmetric tensor field \( B_{\mu\nu} \), dilaton, gravitino, gauge bosons, gauginos, and chiral matter. So the matter states \( (P, s) \) must satisfy

\[ (P + kV)^2 = 2\tilde{c}_k - 2 \sum_j N^L_j \tilde{\phi}_j , \]

\[ (s + k\phi_s)^2 = 2c_k - 2 \sum_j N^R_j \tilde{\phi}_j . \]

Among these massless modes, we are interested in a resulting \( \mathcal{N} = 1 \) SUSY gauge theory. The SUSY condition for orbifold compactification is given by right movers, which are given by four component \( s \), including three component \( \tilde{s} \). When we compactify six dimensions, ten-dimensional supersymmetry generators can be decomposed into \( Q_{(10)} = Q_{(4)} \otimes Q_{(6)} \). The six-dimensional internal space part \( Q_{(6)} \) transforms as \( 4 \) of \( SO(6) \sim SU(4) \). Because the remaining part \( Q_{(4)} \) becomes the four-dimensional generator, the dimension \( 4 \) counts the number of super-symmetries. Its spinorial representation is given by \( |\tilde{s}\rangle = |s_1 s_2 s_3\rangle = |\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle \) with even number of minus signs. Under point group, it transforms as

\[ Q_{(6)} \rightarrow \exp(2\pi i s \cdot \phi) Q_{(6)} . \]

The invariant component corresponds to the unbroken supersymmetry generator. For \( \mathcal{N} \geq 1 \) supersymmetry, we need at least one solution, say if we choose \( s = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \), the argument of exponent vanishes for

\[ \phi_1 - \phi_2 - \phi_3 = 0 \mod Z . \]

The number of solutions, \( \mathcal{N} \), counts the number of unbroken supersymmetry generators from orbifold compactification. Note that for our choice \( \phi_s = (\frac{5}{12}, \frac{4}{12}, \frac{1}{12}) \) the above condition is satisfied only for \( \tilde{s} = (\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \). We can introduce another set, \( \tilde{r} \), of half integers so that entries of \( \tilde{s} + \tilde{r} \) become integers,

\[ \tilde{r} = (r_1, r_2, r_3) . \]

Namely, the SUSY condition for orbifolded spectrum is \( \phi_s \cdot \tilde{r} = 0 \) [7].

### 2.3.1. Gauge multiplet

Gauge boson multiplets appear in the untwisted sector \( U \), satisfying

\[ P^2 = 2, \quad P \cdot V = 0, \quad P \cdot a_i = 0 \quad \text{for all } i , \]

where the first one is the masslessness condition and the second and the third ones are the orbifold conditions. The corresponding right movers, satisfying the mass-shell and orbifold conditions chooses two \( s \)'s with \( s^2 = 1 \), which are always \( CPT \) conjugates of each other. In this way, we obtain the gauge multiplet. As expected, the multiplicity \( (P) \) of gauge bosons is 1.
2.3.2. Matter multiplets

Other massless states can appear in $U$ also, for

\[ P^2 = 2, \quad P \cdot V = \frac{k}{N} \text{ for } k \neq 0 \mod N, \quad P \cdot a_i = 0 \text{ for all } i. \]  

(21)

We combine $\tilde{s}$ and $\tilde{r}$ to distinguish untwisted matter so that they appear in three categories under orbifolding, i.e. differing in sub-lattice shifts,

\[ U_1: \quad \tilde{s} + \tilde{r} = (-1, 0, 0), \quad U_2: \quad \tilde{s} + \tilde{r} = (0, 1, 0), \quad U_3: \quad \tilde{s} + \tilde{r} = (0, 0, 1). \]  

(22)

There are fixed points in field theory orbifolds. In string orbifolds also, we must consider physics related to fixed points. Massless strings can sit at fixed points, which is found by the mass-shell condition at fixed point. But noting that some linear combinations of strings sitting at several fixed points may be taken to satisfy the orbifold condition, we consider twisted sectors. For $\mathbb{Z}_N$ orbifold, we consider $k = 1, \ldots, N - 1$ twisted sectors, $T_k$. The $\mathbb{CP}T$ conjugates of $T_k$ appear in $T_{N-k}$. Thus, in non-prime orbifolds $\mathbb{Z}_4, \mathbb{Z}_6, \mathbb{Z}_8, \mathbb{Z}_{12}$, the sector $T_{N/2}$ contains $\mathbb{CP}T$ conjugates also.

For untwisted matter, multiplicity ($P$) is given just by counting all possible states. Multiplicity in the twisted sector is more involved. The method of linear combination can come with complex numbers. This is taken into account in the (generalized) GSO projector, which projects out non-physical states. It can be read off from the one loop partition function of string [7,31];

\[ P_k = \frac{1}{N} \sum_{l=0}^{N-1} \tilde{\chi}(\theta^k, \theta^l) e^{2\pi i l \Theta_0} \equiv \frac{1}{N} \sum_{l=0}^{N-1} \tilde{\chi}(\theta^k, \theta^l) \Delta^l, \]  

(23)

where $N$ is the order of $\mathbb{Z}_N$ orbifolds, and

\[ \Theta_0 = \sum_j (N_j^L - N_j^R) \hat{\phi}_j - \frac{k}{2} (V^2 - \phi_s^2) + (P + kV) \cdot V - (\tilde{s} + k\phi_s) \cdot \phi_s + \text{integer}, \]  

(24)

where $j$ denotes the coordinates of the 6-dimensional compactified space running over $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$ in complexified coordinates, and $\hat{\phi}_j = \phi_{s_i} \text{sgn}(\hat{\phi}_j)$ where $\text{sgn}(\hat{\phi}_j) = -\text{sgn}(\hat{\phi}_j)$ [29]. It turns out that $N_j^R = 0$ generically for the massless right mover states in our $\mathbb{Z}_{12-1}$ orbifold compactification. The $\tilde{\chi}(\theta^m, \theta^k)$ in Eq. (23) denotes the degenerate factor tabulated in Table 1 [7,32]. For the sectors wound by Wilson lines, the Wilson line modified shift vector $V_f$ is used instead of $V$.

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Degeneracy factor $\tilde{\chi}(\theta^k, \theta^l)$ in the $\mathbb{Z}_{12-1}$ orbifold</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k \times l$</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>27</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>16</td>
</tr>
</tbody>
</table>

For the same chirality, i.e. $L$, we use $(- - -)$ instead of $(+++)$. So, we take $U_1 = (-1, 0, 0)$.
In the presence of a Wilson lines ($\equiv a^I$), the GSO projector Eq. (23) needs to be modified as [29]

$$e^{2\pi il\theta_0} \rightarrow \frac{1}{N_W} \sum_{f=0}^{N_W-1} e^{2\pi il\theta_f},$$

(25)

where $N_W = 3$ for Wilson line of order three in case of $\mathbf{Z}_{12-1}$, and $\Theta$ in Eq. (24) should also be modified as

$$\Theta_f = \sum_j (N^L_j - N^R_j) \hat{\phi}_j - \frac{k}{2} (V_f^2 - \phi_s^2) + (P + kV_f) \cdot V_f - (\bar{s} + k\phi_s) \cdot \phi_s + \text{integer}.$$  

(26)

where $V_f \equiv (V + m_f a_3)$, and Eq. (23) can be rewritten as $P_k = \sum_{f=0}^{N_W-1} P_k(f)$ where

$$P_k(f) = \frac{1}{N_{NW}} \sum_{l=0}^{N-1} \bar{\chi}(\theta^k, \theta^l) e^{2\pi il\theta_f}.$$  

(27)

Note that in the $\mathbf{Z}_{12-1}$ model, $f = \{0, 1, 2\} \equiv \{f_0, f_+, f_-\}$ and $P_k(f_0) = P_k(f_+) = P_k(f_-)$ for $k = 3, 6, 9$.

3. $\mathbf{Z}_{12-1}$ model

We choose the following shift vector and Wilson lines

$$V = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 5 & 6 & 12 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(28)

$$a_3 = a_4 = \begin{pmatrix} 0 & 5 & 0 \\ -1 & -1 & 1 \\ \frac{1}{3} & \frac{1}{3} & 3 \\ 0 & 0 & 0 \end{pmatrix},$$

(29)

$$a_1 = a_2 = a_5 = a_6 = 0,$$

which satisfies Eq. (7): it gives $V^2 - \phi_s^2 = \frac{1}{2}$ modular $2 \cdot \text{(integer)}/12$. Since the Wilson line is a $\mathbf{Z}_3$ shift, it distinguishes three cases. These satisfy the following conditions [31,33]:

$$12(\|V\|^2 - |\phi_s|^2) = 0 \mod \text{even integer},$$

$$12(V \cdot a_3) = 0 \mod \text{integer},$$

$$12|a_3|^2 = 0 \mod \text{even integer}.$$  

(30)

Thus, we consider the following effective shifts distinguishing twisted sector $T_k$ to $T_k^0, T_k^+, T_k^-$ ($k \neq 0, 3, 6, 9$) by

$$V_0 = V,$$

(31)

$$V_+ = V + a_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 5 & 2 & 4 & 8 \\ 4 & 4 & 4 & 4 & 4 & 4 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 2 & 8 & 0 \end{pmatrix},$$

(32)

$$V_- = V - a_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 5 & 10 & -4 \end{pmatrix} \begin{pmatrix} 2 & 2 & -8 & 0 \end{pmatrix}.$$  

(33)
For future convenience, we list
\[ V_0^2 - \phi_s^2 = \frac{1}{2} , \]
\[ V_+^2 - \phi_s^2 = \frac{5}{6} , \]
\[ V_-^2 - \phi_s^2 = \frac{3}{2} . \]  
(34)

### 3.1. Gauge group from untwisted sector

For the gauge multiplet, we search for roots satisfying
\[ P^2 = 2, \quad P \cdot V = 0 \quad \text{and} \quad P \cdot a_3 = 0 , \]
and obtain the following unbroken gauge group
\[ [ SU(5) \times U(1)_X \times U(1)_Z ] \times [ SU(2) \times SO(10) \times U(1)_1 ] ]' \]
where we choose the \( U(1)_X \) of flipped \( SU(5) \) as
\[ Q_X = (-2, -2, -2, -2, -2; 0, 0, 0)(0^8)' . \]  
(36)

For example, one can see that the following is the roots of \( SU(5) \):
\[ (1 -1 0 0 0; 0 0 0) \quad \text{nonzero roots among 24 of} \; SU(5), \]  
(37)

where the underlined entries allow permutations.

### 3.2. Matter from untwisted sector

#### 3.2.1. Chirality

The chirality is determined by the 8 component \( SO(8) \) spinor of the Ramond sector of right movers. It is labeled by \( s = (s_0, \tilde{s}) = \{ \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2} \} \) with an even number of minus signs.

We define the 4D chirality \( \chi \) as the one originating from the first entry of \( s \) denoted by \( \oplus \) or \( \ominus \), i.e.
\[ s = \{ \oplus \text{ or } \ominus, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2} \} = \left\{ \frac{1}{2} \chi, \tilde{s} \right\} , \]  
(38)
\[ \chi = 2s_0 = 2\oplus \text{ or } 2\ominus . \]  
(39)

Let us call \( \chi = 1 \) \(( -1 )\) as ‘right- (left-)handed’, and \( \tilde{s} \) has three components in terms of \( \pm \frac{1}{2} \).

In the untwisted sector \( U \), we have \( U_1, U_2 \) and \( U_3 \) as defined in Eq. (22). These \( U_i \) correspond to the untwisted \( \tilde{s} = (---), (++-), (+--), \) respectively [or to \((+++), (-+-), (--+)\), respectively, since antiparticles can be used also].

The chirality in the twisted sector can be similarly defined by the 8 component \( SO(8) \) spinor of the Ramond sector of right movers.

#### 3.2.2. Spectrum

The massless matter fields are those with \( P \cdot V \neq 0 \) and satisfy the masslessness condition. \( P \) must satisfy \( P^2 = 2, \; P \cdot a_3 = \text{integer} \) and \( P \cdot V = \frac{k}{12} \) where \( k = 1, 2, \ldots, 11 \). We will consider \( k = 1, 2, \ldots, 6 \) only, since the rest will provide their \( \mathcal{CTP} \) conjugates. Here, it is sufficient to look at 3 cases only, \( k = 1, 4, 5 \), to keep the GSO allowed states.
We have the convention that the highest weight of the complex conjugated representation is in fact the lowest one so that all the weights of the complex conjugated representation are obtained by adding simple roots.

Let $\alpha$ denote the phase $e^{2\pi i/12}$. For $k = 1$ or $P \cdot V = \frac{1}{12}$, the left movers obtain a phase $\alpha$. We need an extra phase $\alpha^{-1}$ from the right movers, which is accomplished by $e^{-2\pi is\phi_s}$ where $\phi_s = \frac{1}{12}(5\ 4\ 1)$ and $s = (\Theta+++)$. It is left-handed and allows $U_3$. For $k = 4$, $s = (\Theta+++) \propto \alpha^{-4}$; thus it is left-handed and provides $U_2$. For $k = 5$, $s = (\Theta+++) \propto \alpha^{-5}$; thus it right-handed and provides $U_1$. $\alpha$ from the right movers is provided by $s = (\Theta--+) \propto \alpha^{-k+11}$ which thus will couple to $k = 11$. It is right-handed. $\alpha^4$ from the right movers is provided by $s = (\Theta--+) \propto \alpha^{-k+11}$ which will couple to $k = 8$. It is right-handed. $\alpha^5$ from the right movers is provided by $s = (\Theta--+) \propto \alpha^{-k+11}$ which will couple to $k = 7$. It is left-handed. These particles for $k > 6$ give the antiparticle spectra. The chiralities and $U_i$s are shown in Tables 2 and 3.

Thus, the vectors for $p \cdot V = \frac{1}{12}$, $\frac{4}{12}$, and $\frac{7}{12}$ give

$$
\begin{align*}
(1_5 + 5_3 + 10_{-1})^L_{U_3}, & \quad (\bar{5}_2)^L_{U_2}, & \quad (1_5 + 5_3 + 10_{-1})^L_{U_1}, & \quad (10)^L_{U_2}
\end{align*}
$$

and their $CT\mathcal{P}$ conjugates appear in $p \cdot V = \frac{11}{12}$, $\frac{8}{12}$, and $\frac{5}{12}$. Here we listed only $U(1)_X$ charges as boldfaced subscripts. Note that for the untwisted $k = 6$ sector there is no way to provide an additional $\alpha^6$ at the massless level from the right movers.

### 4. Twisted sectors with Wilson line

The $\mathbb{Z}_{12-1}$ with the twist vector $\phi = \frac{1}{12}(5\ 4\ 1)$ has three fixed points in the second torus for the prime order $\theta^1$ and $\theta^3$ twists. This is because it is the same as $\mathbb{Z}_3$. For the first and the third torus the origin is the only fixed point, viz. Fig. 1. For the other twists such as $k = 4$ and 6, counting the number of massless states involves a more complicated nonvanishing projection operator $P_{\phi^k}$. But addition of Wilson lines can distinguish some fixed points. In fact, for $\mathbb{Z}_{12-1}$ possible Wilson lines must satisfy $3a_3 = 0$, $3a_4 = 0$ and $a_i = 0$ ($i \neq 3, 4$) so that any combination
of $k(V + m_f a_i)$ is another shift vector. Since $Z_{12^{-1}}$ allows $3a_3 = 0$ modular integer, the Wilson line is a $Z_3$ shift.

In the second torus, Wilson lines must be symmetric, $a_3 = a_4$. Then, the $k$th twisted sector is distinguished by $kV$, $k(V + a_3)$, and $k(V - a_3)$, which is denoted as $\tilde{V}$,

$$\tilde{V} = k(V + m_f a_3), \quad m_f = 0, \pm 1,$$

(41)

or

$$\tilde{V} = k\{V_0, V_+, V_\}.$$

(42)

Since $3a_3 = 6a_3 = 0$, $2V_\pm$ and $5V_\mp$ in the $T_2$ and $T_3$ sectors are equivalent to $2V \mp a_3$ and $5V \pm a_3$, respectively. On the other hand, $4V_\pm$ in the $T_4$ sector is equivalent to $4V \pm a_3$.

The masslessness condition is

$$(P + \tilde{V})^2 = 2\tilde{c}_k - 2 \sum_j N^L_{f_j} \tilde{\phi}_j.$$  

(43)

For the $\theta^k$ twist ($k = 1, 2, \ldots, 6$), we have

$$2\tilde{c}_k = \begin{cases} 
\frac{210}{144}, & k = 1; \\
\frac{192}{144}, & k = 4; \\
\frac{216}{144}, & k = 2; \\
\frac{210}{144}, & k = 5; \\
\frac{234}{144}, & k = 3; \\
\frac{216}{144}, & k = 6,
\end{cases}$$

(44)

for the left movers, and

$$2c_k = \begin{cases} 
\frac{11}{24}, & k = 1; \\
\frac{1}{3}, & k = 4; \\
\frac{1}{2}, & k = 2; \\
\frac{11}{24}, & k = 5; \\
\frac{5}{6}, & k = 3; \\
\frac{1}{2}, & k = 6,
\end{cases}$$

(45)

for the right movers.

For the sectors wound by Wilson lines, $V_\pm$ are used instead of $V_0$. The untwisted sector $k = 0$ and twisted sectors for $k = 3, 6, 9$ are not affected by Wilson lines since the Wilson line condition, $3a_3 = 0$, makes it trivial. So, for $k = 3, 6, 9$, there is the additional condition, $(P + kV)$ $\cdot$ $a_i = 0$, which is applicable to $T_6$ only in our case. For $k \neq 3, 6, 9$, the multiplicity for each twisted sector $k(V + m_f a_3)$ is $\mathcal{P} = \frac{1}{3} \mathcal{P}_k$.

The formula for multiplicity, Eq. (23), is the GSO allowed number of states.\(^5\) For non-prime orbifolds such as $Z_{12}$, the multiplicity (23) is nonvanishing even if $\Delta$ were not 1. Only for those with pure $Z_{12}$ twists, i.e. $k = 1, 2$ and 5, the multiplicity is counted by those with the vanishing phase.

The twisted sectors for $k = 3, 6, 9$ are not affected by the additional Wilson lines since $3a_3 = 0$. Note that the untwisted sector also is not distinguished by Wilson lines, but Wilson lines give the modular invariance condition $P \cdot a_i = 0$ in the untwisted sector. By the same token, in the sectors where $3a_3 = 0$ ($k = 0, 3, 6, 9$, with 0 corresponding to the untwisted sector), the modular invariance condition restricts Wilson lines [34],

$$(P + kV) \cdot a_3 = 0 \mod Z, \quad k = 0, 3, 6, 9.$$  

(46)

\(^5\) For the prime orbifold $Z_3$, the multiplicity is just $\frac{1}{2}(1 + \Delta + \Delta^2)$ which can be either 1 for $\Delta = 1$ or 0 for $\Delta = e^{\pm 2\pi i/3}$. So in $Z_3$ it is sufficient to count those with the vanishing phase, i.e. $(P + V) \cdot V - (s + \phi) \cdot \phi = 0$. It is so also in the $k$th twisted sector of $Z_{12^{-1}}$ if $\chi(\theta^k, \theta^l)$ are the same for all $l$.\]
Table 4
Left-handed massless states satisfying \((P + 6V) \cdot a_3 = 0 \mod Z\) in \(T_6\)

<table>
<thead>
<tr>
<th>(P + 6V)</th>
<th>((N^L)_j)</th>
<th>(\Theta_0)</th>
<th>(\mathcal{P}_6)</th>
<th>(\chi)</th>
<th>Labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>(53; \frac{1}{2}, 0, 0)(0^8)')</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>2</td>
<td>(L)</td>
<td>(T_{65})</td>
</tr>
<tr>
<td>(10_{-1}; \frac{1}{2}, 0, 0)(0^8)')</td>
<td>0</td>
<td>0</td>
<td>4</td>
<td>(L)</td>
<td>(T_{610})</td>
</tr>
<tr>
<td>(1_{-5}; \frac{1}{2}, 0, 0)(0^8)')</td>
<td>0</td>
<td>(\frac{1}{2})</td>
<td>2</td>
<td>(L)</td>
<td>(T_{611})</td>
</tr>
<tr>
<td>(\bar{5}_3; -\frac{1}{2}, 0, 0)(0^8)')</td>
<td>0</td>
<td>(-\frac{1}{6})</td>
<td>2</td>
<td>(L)</td>
<td>(T_{63})</td>
</tr>
<tr>
<td>(10_{1}; -\frac{1}{2}, 0, 0)(0^8)')</td>
<td>0</td>
<td>(\frac{1}{3})</td>
<td>3</td>
<td>(L)</td>
<td>(T_{610})</td>
</tr>
<tr>
<td>(1_{5}; -\frac{1}{2}, 0, 0)(0^8)')</td>
<td>0</td>
<td>(-\frac{1}{6})</td>
<td>2</td>
<td>(L)</td>
<td>(T_{611})</td>
</tr>
<tr>
<td>(1_{0}; \frac{1}{2}, \frac{1}{2}) (0^8)')</td>
<td>(l_3)</td>
<td>0</td>
<td>4</td>
<td>(L)</td>
<td>(h_1)</td>
</tr>
<tr>
<td>(\bar{1}_{3})</td>
<td>(-\frac{1}{6})</td>
<td>2</td>
<td>(L)</td>
<td>(\bar{h}_2)</td>
<td></td>
</tr>
<tr>
<td>(l_1)</td>
<td>(\frac{1}{2})</td>
<td>2</td>
<td>(L)</td>
<td>(h_3)</td>
<td></td>
</tr>
<tr>
<td>(l_1)</td>
<td>(\frac{1}{3})</td>
<td>3</td>
<td>(L)</td>
<td>(h_4)</td>
<td></td>
</tr>
<tr>
<td>(1_{0}; -\frac{1}{2}, -\frac{1}{2}(0^8)')</td>
<td>(l_3)</td>
<td>(\frac{1}{2})</td>
<td>2</td>
<td>(L)</td>
<td>(\tilde{h}_1)</td>
</tr>
<tr>
<td>(\bar{l}_3)</td>
<td>(\frac{1}{3})</td>
<td>3</td>
<td>(L)</td>
<td>(\tilde{h}_2)</td>
<td></td>
</tr>
<tr>
<td>(l_{1})</td>
<td>0</td>
<td>4</td>
<td>(L)</td>
<td>(\tilde{h}_3)</td>
<td></td>
</tr>
<tr>
<td>(l_{1})</td>
<td>(-\frac{1}{6})</td>
<td>2</td>
<td>(L)</td>
<td>(\tilde{h}_4)</td>
<td></td>
</tr>
</tbody>
</table>

But other twisted sectors are not affected, in particular the \(k = 1, 2, 4, 5\) sectors. For these sectors, the multiplicity for each of \(k(V + m_f a_3)\) is \(\frac{1}{2}P_{6k}\) where \(P_{6k}\) is given in Eq. (23).

Let us present twisted sectors for \(k = 6, 1, 2, 3, 4, 5\) in order. The \(k = 6\) twisted sector \(T_6\) contains the \(CT\) \(\mathcal{P}\) conjugates in \(T_6\) again. The spectra in the \(k = 1, \ldots, 5\) twisted sectors accompany their \(CT\) \(\mathcal{P}\) conjugates in \(k = 11, \ldots, 7\).

4.1. Twisted sector \(T_6\)

The massless condition for the left mover is \(\frac{1}{2}|P + 6V|^2 + \sum_j (N^L)_j(\tilde{\phi})_j = \frac{3}{4}\), where \(P\) is the \(E_8 \times E'_8\) weight vectors. The left mover states satisfying the massless condition always appear vector-like in the \(T_6\) sector. In general, however, they carry different phases from those of the counterpart states with opposite quantum numbers. Thus, chiral matter spectrum is possible even in \(T_6\) after imposing the GSO projection by Eq. (23). In view of (46), we additionally require

\[
(P + 6V) \cdot a_3 = 0 \mod Z. \tag{47}
\]

The massless states with left-handed chirality satisfying this constraint, and their multiplicity numbers determined by \(\mathcal{P}_6\) are listed in Table 4.

For simplicity, here we employed the following abbreviations for the \(SO(10)\) spinors and neutral singlets under \(SU(5) \times U(1)_X\),

\[
\begin{align*}
5_3 & \equiv (+---), \quad \overline{10}_{-1} \equiv (++++), \quad 1_{-5} \equiv (++++++), \tag{48} \\
\bar{5}_{-3} & \equiv (-++++)\), \quad 10_1 \equiv (-+++), \quad 1_{5} \equiv (------), \tag{49} \\
1_0 & \equiv (0, 0, 0, 0, 0). \tag{50}
\end{align*}
\]
From the $T_6$ sector, thus we have the following massless states,

\[
\mathbf{10}_{-1}^L + 3\{\mathbf{10}_1^L, \bar{\mathbf{10}}_{-1}^L\} + 2\{\mathbf{5}_{-3}^L, \mathbf{5}_3^L\} + 2\{\mathbf{1}_5^L, \mathbf{1}_{-5}^L\} + 22 \text{ neutral singlets} + CTP \text{ conjugates.}
\]

(51)

While the multiplicity number for $\mathbf{10}_{-1}^L$ is 4, the multiplicity of $\mathbf{10}_1^L$ is 3. One of $\mathbf{10}_{-1}^L$ s provides one generation of the MSSM matter, \{Q, d^c, \nu^c\}. The remaining vector-like pairs of $\{\mathbf{10}_1^L, \bar{\mathbf{10}}_{-1}^L\}$ could be utilized to break $SU(5) \times U(1)_X$ into the MSSM gauge group.

We present the calculation in detail for the first row. The $\theta_6$ twist vectors are

\[
\tilde{\phi} = 6\phi_s = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{pmatrix},
\]

\[
\tilde{V} = 6V \equiv \left( \begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).
\]

(52)

In the $\theta_6$ twisted sector, consider $P = (0^8)(0^8)$. With some shifts, $P$ can be $(\{[-1]^5; -2, -3, 0\}) \times (\{-1, -1; -6, 0^5\})$. Then, from (43) and (44) we require $(P + 6V)^2 = 2\tilde{c}_6 = \frac{216}{144} = \frac{3}{2}$ which is certainly satisfied with $\tilde{N}_L = 0$. Therefore, the state $P = (0^8)(0^8)$ is massless. The GSO projection is given when combined with the right movers. The masslessness condition for right movers is $(s + 6\phi)^2 = \frac{1}{2}$, from which we will determine the chirality.

Let us calculate the masslessness condition of the left movers first, the multiplicity and the chirality.

The masslessness condition for left movers becomes

\[
(P + \tilde{V})^2 + 2 \sum_j N_j^L \tilde{\phi}_j = \frac{3}{2}.
\]

(53)

Thus, the vectors satisfying Eq. (53) with Eq. (52) constitute the representation $\bar{\mathbf{5}}_L$. These weights are

- $\mathbf{15}$: (0 0 0 0 0 0 0 0),
- $\bar{\mathbf{5}}_{-3}$: (−1 0 0 0 0 −1 0 0),
- $\mathbf{10}_1$: (−1 −1 −1 0 0 −1 0 0),

(54)

where we calculated $U(1)_X$ charges as $(P + \tilde{V}) \cdot Q_X$. Their $CTP$ conjugates are

\[
\begin{align*}
(−1 & −1 −1 −1 −1 −1 0 0), \\
(−1 & −1 −1 −1 0 0 0 0), \\
(−1 & −1 0 0 0 0 0 0).
\end{align*}
\]

(55)

Note that for $P = (0^8)(0^8)$,

\[
(P + 6V) \cdot V = \frac{5}{6}, \quad \tilde{\phi} \cdot \phi_s = \frac{1}{4}.
\]

(56)

To obtain the chirality $\chi$, we look at $s = (\frac{1}{2} \chi, \tilde{s})$ allowing nonvanishing multiplicities. Then the phase of $\Delta_{\theta^6}$ is found as $(P + \tilde{V}) \cdot V - (\tilde{s} + \tilde{\phi}) \cdot \phi_s - \frac{1}{2} k(V^2 - \phi_s^2)$. With $k = 6$, we have

\footnote{See Appendix D of [7].}
The multiplicity for \( s^2 = 1 \) and \((P + 6V) \cdot V = \frac{1}{k} \cdot s \) and \(-\frac{1}{k} \cdot s \), respectively, and \( \oplus (\ominus) \) is \( R \)-\( (L) \)-handed. Note that \((P + \tilde{V}) \cdot V = -\frac{1}{2}k(V^2 - \phi_s^2) = \frac{1}{3} \).

<table>
<thead>
<tr>
<th>( \frac{1}{2} \chi )</th>
<th>( \tilde{s} = (r + \omega) )</th>
<th>( \tilde{s} \cdot \phi_s )</th>
<th>( \Delta ) phase</th>
<th>Multiplicity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \oplus )</td>
<td>++ +</td>
<td>( \frac{5}{12} )</td>
<td>-( \frac{4}{12} \cdot 2\pi )</td>
<td>3</td>
</tr>
<tr>
<td>( \ominus )</td>
<td>+ +</td>
<td>0</td>
<td>( \frac{1}{12} \cdot 2\pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \ominus )</td>
<td>+</td>
<td>( -\frac{1}{12} )</td>
<td>( \frac{2}{12} \cdot 2\pi )</td>
<td>2</td>
</tr>
<tr>
<td>( \ominus )</td>
<td>- +</td>
<td>( -\frac{4}{12} )</td>
<td>( \frac{5}{12} \cdot 2\pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \ominus )</td>
<td>-</td>
<td>( -\frac{5}{12} )</td>
<td>( \frac{6}{12} \cdot 2\pi )</td>
<td>2</td>
</tr>
<tr>
<td>( \ominus )</td>
<td>+ +</td>
<td>0</td>
<td>( \frac{1}{12} \cdot 2\pi )</td>
<td>0</td>
</tr>
<tr>
<td>( \ominus )</td>
<td>+</td>
<td>( \frac{1}{12} )</td>
<td>0 \cdot 2\pi</td>
<td>4</td>
</tr>
</tbody>
</table>

For \( s^2 = \frac{1}{2} \), the phase of \( \Delta \) is \( 2\pi \) times (\( \frac{1}{12} \) - \( \tilde{s} \cdot \phi_s \)), and hence we obtain the multiplicity listed in Table 5. Then, we read the chirality \( \chi \), or \( \oplus \) or \( \ominus \), from the first entry of \( s = (\frac{1}{2} \chi | s_1 s_2 s_3) \) if that chirality is allowed by the right mover condition, which is shown in the first column of Table 5. If the first column does not satisfy the right mover condition, we should search for higher \( s_0 \), which will be discussed shortly. The components of the vector \( \tilde{s} \) are the last three \( \pm 1 \)'s of \( s \). The number of massless states are given by \( P_{\theta^6} \). For these, we use the Euler numbers \( \tilde{\chi}(\theta^k, \theta^l) \) given in Table 1 [7]. Therefore, we obtain

\[
P_6 = P_{\theta^6} = \frac{1}{12} \left\{ (1 + \Delta^6)(16 + 4\Delta^3) + \Delta(1 + \Delta)(1 + \Delta^3 + \Delta^6 + \Delta^9) \right\}
\]

which becomes 4, 2, 2, and 3 for \( \Delta = 1, -1, \Delta = -1, +1 \), respectively, and 0 for the other cases. We know that \( s \cdot \tilde{\phi} = s \cdot \tilde{\phi} + \) (integer). Now consider the right mover condition. For \( s^2 = 1 \), the masslessness condition \( (s + \tilde{\phi})^2 = \frac{1}{2} \) leads to \( \tilde{s} = (-\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}) \) where we used the shifted \( \tilde{\phi} \) of Eq. \( (52) \). The relevant ones appear in the third and fifth rows of Table 5. Among these one set is the \( CTP \) conjugates of the other. The \( U(1)_X \) charge is \( (P + 6V) \cdot Q_X = 5 \). Thus, we obtain two singlets as shown in the third row of Table 4.

Consider \( P = (-1 \ 0 \ 0 \ 0 \ 0 \ -1 \ 0 \ 0) (0^8) \) and \( P = (-1 \ -1 \ -1 \ -1 \ 0 \ 0 \ 0 \ 0) (0^8) \). For \( P = (-1 \ 0 \ 0 \ 0 \ 0 \ -1 \ 0 \ 0) (0^8) \),

\[
(P + 6V) \cdot V = \frac{1}{6}, \quad \tilde{\phi} \cdot \phi_s = \frac{1}{4}.
\]

For \( k = 6 \), the phase becomes \( (P + \tilde{V}) \cdot V = (\tilde{s} + \tilde{\phi}) \cdot \phi_s - \frac{1}{2}k(V^2 - \phi_s^2) = (-\frac{7}{12} \tilde{s} \cdot \phi_s). \) We add \( -\frac{8}{12} \cdot 2\pi \) to the fourth column entries of Table 5. The masslessness condition chooses \( \tilde{s} = (-\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}) \), the third and fifth rows again, leading to the multiplicity 2. \( (-1 \ 0 \ 0 \ 0 \ 0 \ -1 \ 0 \ 0) \) is \( \tilde{s} \) whose \( U(1)_X \) charge is 3. For \( P = (-1 \ -1 \ -1 \ -1 \ 0 \ 0 \ 0 \ 0) (0^8) \),

\[
(P + 6V) \cdot V = -\frac{1}{6}, \quad \tilde{\phi} \cdot \phi_s = \frac{1}{4}.
\]

Now the phase becomes \( (P + \tilde{V}) \cdot V = (\tilde{s} + \tilde{\phi}) \cdot \phi_s - \frac{1}{2}k(V^2 - \phi_s^2) = (-\frac{11}{12} \tilde{s} \cdot \phi_s) = (-\frac{1}{12} - \tilde{s} \cdot \phi_s). \) Thus we obtain the fourth column entries of Table 5. The masslessness condition chooses \( \tilde{s} = (-\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2}) \), the third and fifth rows again, leading to the multiplicity 2.
Thus we obtain the first and fourth row entries of Table 4.

Next consider $\tilde{10}$s and $10$s. So consider $P = (-1 - 1 0 0 0 0 0 0)(0^8)$ and $P = (-1 - 1 - 1 0 0 - 1 0 0)(0^8)$. For $P = (-1 - 1 0 0 0 0 0 0)(0^8)$, we have

\[
(P + 6V) \cdot V = \frac{1}{3}, \quad \tilde{\phi} \cdot \phi_s = \frac{1}{4}.
\]

(62)

The phase becomes $(P + \tilde{V}) \cdot V - (\tilde{\phi} + \phi_s) \cdot \phi_s \cdot \frac{1}{2} k (V^2 - \phi_s^2) = (\frac{\tilde{5}}{12} - \tilde{\phi} \cdot \phi_s) = (\frac{7}{12} - \tilde{\phi} \cdot \phi_s)$. We add $\frac{6}{12} \cdot 2\pi$ to the fourth column entries of Table 5. The masslessness condition for right movers chooses $\tilde{s} = (-\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2})$, the third and fifth rows again, leading to the multiplicity 3 and 4, respectively. For $P = (-1 - 1 - 1 0 0 - 1 0 0)(0^8)$, we have

\[
(P + 6V) \cdot V = -\frac{1}{3}, \quad \tilde{\phi} \cdot \phi_s = \frac{1}{4}.
\]

(63)

The phase becomes $(P + \tilde{V}) \cdot V - (\tilde{\phi} + \phi_s) \cdot \phi_s \cdot \frac{1}{2} k (V^2 - \phi_s^2) = (\frac{1}{12} - \tilde{\phi} \cdot \phi_s) = (\frac{11}{12} - \tilde{\phi} \cdot \phi_s)$. We add $-\frac{2}{12} \cdot 2\pi$ to the fourth column entries of Table 5. The masslessness condition chooses $\tilde{s} = (-\frac{1}{2}, \pm \frac{1}{2}, -\frac{1}{2})$, the third and fifth rows again, leading to the multiplicity 4 and 3, respectively. These four cases are shuffled to list the $CT\bar{P}$ conjugates together,

\[
10_{L} \begin{cases}
10: & (P + 6V) = (-\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} 0 0), \; \Theta, \; \mathcal{P}_6 = 3, \; Q_X = -1, \\
\bar{10}: & (P + 6V) = (-\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} 0 0), \; \Theta, \; \mathcal{P}_6 = 3, \; Q_X = 1,
\end{cases}
\]

(64)

\[
10_{L} \begin{cases}
10: & (P + 6V) = (-\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} 0 0), \; \Theta, \; \mathcal{P}_6 = 4, \; Q_X = 1, \\
\bar{10}: & (P + 6V) = (-\frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - \frac{1}{2} 0 0), \; \Theta, \; \mathcal{P}_6 = 4, \; Q_X = -1.
\end{cases}
\]

(65)

Thus we obtain the second and fifth row entries of Table 4.

Other singlets with nonvanishing oscillators are also allowed, which are shown in Table 4.

4.2. Twisted sector $T_1$

The massless condition for the right mover in the $T_1$ sector is $(s + \phi_s)^2 = \frac{11}{24}$. It allows only one right-handed state $s = (\Theta -- -)$, which gives $(\tilde{s} + \phi_s) \cdot \phi_s = -\frac{1}{8}$.

Since $-\frac{k}{2} (V^2 - \phi_s^2) = -\frac{1}{4}$ for $k = 1$, the phase in Eq. (24) is given by $\Theta_0 = (P + V) \cdot V + \sum j(N^L), \tilde{\phi} j - \frac{1}{8}$.

The $T_1$ sector is distinguished by Wilson lines: $V_0 = V$, $V_+ = V + a_1$, and $V_- = V - a_1$. These sectors are denoted as $T_1^0$, $T_1^+$ and $T_1^-$. 

\((-1 - 1 - 1 - 1 0 0 0 0 0)\) is 5 whose $U(1)_Y$ charge is $-3$. The above four cases are shuffled to list the $CT\bar{P}$ conjugates together,
Note the degeneracy factor $\tilde{\chi}(\theta^k, \theta^l)$ for $k = 1, 2, 5$ in Table 1. They take the same value 3 along each horizontal line, as in the prime orbifolds such as $\mathbb{Z}_3$ and $\mathbb{Z}_7$. Thus, only the states with vanishing phase turn out to survive the projection by Eq. (23) in the $T_1$, $T_2$, $T_5$ (and also in $T_{11}$, $T_{10}$, $T_7$) sectors.

The masslessness condition for the left movers (43) gives

$$ (P + kV)^2 = -2 \sum_j N^L_j \tilde{\phi}_j + 2 \tilde{c}_k. \quad (66) $$

For $k = 1$ we have $2\tilde{c}_1 = \frac{35}{24}$. The states satisfying the massless condition and $(P + V_0) \cdot V_0 + \sum_j (N^L_j) (\hat{\phi})_j = \frac{1}{8}$ for $T_1^0$ are listed in Table 6. The multiplicity 3 reduces to 1 due to the distinction by Wilson lines: $V_0$, $V_+$, and $V_-$. In the $T_1^+$ sector with $V_+ = V + a_3$, only the states with $(P + V_+) \cdot V_+ + (N^L_j) (\hat{\phi})_j = \frac{7}{24}$ survives the GSO projection by Eq. (23), which are listed in the middle part of Table 6.

In the $T_1^-$ sector with $V_- = V - a_3$, only the states with $(P + V_-) \cdot V_- + (N^L_j) (\hat{\phi})_j = \frac{5}{8}$ survives the GSO projection by Eq. (23), which are listed in the lower part of Table 6.

The $CT\bar{P}$ conjugates of $T_1$ appear in $T_{11}$.

<table>
<thead>
<tr>
<th>$P + V$</th>
<th>$(N^L_j)$</th>
<th>$P_1(f_0)$</th>
<th>$\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\bar{5}_{-1/2}: \frac{-7}{12}, \frac{6}{12}, 0)(\frac{1}{6}, \frac{1}{6}, 0^6)'$</td>
<td>0</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(\bar{5}_{-1/2}: \frac{-5}{12}, \frac{-6}{12}, 0)(\frac{1}{6}, \frac{1}{6}, 0^6)'$</td>
<td>13</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(\bar{5}_{1/2}: \frac{-1}{12}, 0, \frac{6}{12})(\frac{1}{6}, \frac{1}{6}, 0^6)'$</td>
<td>23</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{-7}{12}, \frac{6}{12}, 0)(\frac{1}{6}, \frac{1}{6}, 0^6)'$</td>
<td>33</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{-5}{12}, \frac{6}{12}, 0)(\frac{1}{6}, \frac{1}{6}, 0^6)'$</td>
<td>12, 43</td>
<td>1 + 1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{1}{12}, 0, \frac{6}{12})(\frac{1}{6}, \frac{1}{6}, 0^6)'$</td>
<td>0</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{-1}{12}, 0, \frac{-6}{12})(\frac{1}{6}, \frac{1}{6}, 0^6)'$</td>
<td>$1, 5_3, {1_2 + 1_3}$</td>
<td>1 + 1 + 1</td>
<td>$L$</td>
</tr>
<tr>
<td>$P + V_+$</td>
<td>$(N^L_j)$</td>
<td>$P_1(f_+)$</td>
<td>$\chi$</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{-5}{12}, \frac{2}{12}, \frac{7}{12})(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}, 0^5)'$</td>
<td>0</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{-5}{12}, \frac{2}{12}, \frac{7}{12})(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}, 0^5)'$</td>
<td>23</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{-1}{12}, \frac{-4}{12}, \frac{-7}{12})(\frac{-5}{6}, \frac{1}{6}, \frac{-1}{3}, 0^5)'$</td>
<td>13</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{-1}{12}, \frac{-4}{12}, \frac{-7}{12})(\frac{1}{6}, \frac{1}{6}, \frac{2}{3}, 0^5)'$</td>
<td>33</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$P + V_-$</td>
<td>$(N^L_j)$</td>
<td>$P_1(f_-)$</td>
<td>$\chi$</td>
</tr>
<tr>
<td>$(5_{1/2}: \frac{-1}{12}, \frac{4}{12}, \frac{2}{12})(\frac{1}{6}, \frac{1}{6}, \frac{-2}{3}, 0^5)'$</td>
<td>0</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{-5}{12}, \frac{-2}{12}, \frac{8}{12})(\frac{1}{6}, \frac{1}{6}, \frac{-2}{3}, 0^5)'$</td>
<td>0</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{-7}{12}, \frac{-2}{12}, \frac{4}{12})(\frac{1}{6}, \frac{1}{6}, \frac{-2}{3}, 0^5)'$</td>
<td>13</td>
<td>1</td>
<td>$L$</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{-1}{12}, \frac{-8}{12}, \frac{2}{12})(\frac{1}{6}, \frac{1}{6}, \frac{-2}{3}, 0^5)'$</td>
<td>13</td>
<td>1</td>
<td>$L$</td>
</tr>
</tbody>
</table>
### 4.3. Twisted sector $T_2$

The massless condition for the right mover in the $T_2$ sector is $(s + 2\phi_s)^2 = \frac{1}{4}$. It allows only one right-handed state $s = (\Theta^{-\cdots})$, which gives $(s + 2\phi_s) \cdot \phi_s = \frac{1}{6} = -\frac{3}{6}$. Since $-\frac{k}{3}(V^2 - \phi^2) = -\frac{1}{2}$ for $k = 2$, the phase of $\Delta$ in Eq. (24) is given by $\Theta_0 = (P + 2V) \cdot V + \sum_j (N^L)_j (\phi_j + \frac{1}{2})$. The $T_2$ sector is also distinguished by Wilson lines: $V_0 = V$, $V_+ = V + a_3$, and $V_- = V - a_3$. These sectors are denoted as $T_2^0$, $T_2^+$ and $T_2^-$.

Since $\tilde{\chi}(\theta^k, \theta^l)$ with $k = 2$ in Eq. (23) takes the same value 3 along the horizontal line, only the states with $\Theta_{0, \pm} = 0$ survive the projection operator in the $T_2$ sector. In the $T_2^0$ sector, hence, the massless states satisfying the condition $(P + 2V) \cdot V + \sum_j (N^L)_j (\phi_j) = -\frac{1}{3}$ are selected. Similarly, in the $T_2^+$ and $T_2^-$ sectors, $(P + 2V_+) \cdot V_+ + \sum_j (N^L)_j (\phi_j) = 0$ and $(P + 2V_-) \cdot V_- + \sum_j (N^L)_j (\phi_j) = -\frac{1}{3}$ should be chosen. The massless condition for the left mover is $|P + 2V_{(\pm)}|^2 + 2\sum_j (N^L)_j (\phi_j) = \frac{3}{2}$. The allowed shifted $E_8 \times E_8$ weight vectors $(P + 2V)$ in the $T_2$ sector are shown in Table 7.

Consider the first row of Table 7. Since

$$2V = \left(\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 5 \\ 2 & 2 & 2 & 2 & 2 & \end{array}\right) \left(\begin{array}{c} 1 \\ 3 \\ 3 \\ 0^6 \end{array}\right)^T,$$

Table 7

| Chiral matter fields satisfying $\Theta_0 = 0$, $\Theta_+ = 0$, and $\Theta_- = 0$ in the $T_2^0$, $T_2^+$, and $T_2^-$ sectors, respectively. Here $I_0 \equiv (0, 0, 0, 0, 0)$ and the hidden sector $16'$ and $10'$ are not bold-faced not to be confused with observable sector representations. There are four $SU(2)'$ doublets of $D_{I_2}^{T_2}$ |
|---|---|---|---|
| $P + 2V$ | $(N^L)_j$ | $\mathcal{P}_2(f_0)$ | $\chi$ | Labels |
| $(53; \frac{-1}{6}, 0^2)(\frac{1}{3}, \frac{1}{3}; 0^6)'$ | 0 | 1 | $L$ | $T_{25}$ |
| $(1-5; \frac{-1}{6}, 0^2)(\frac{1}{3}, \frac{1}{3}; 0^6)'$ | 0 | 1 | $L$ | $T_{21}$ |
| $(10; \frac{1}{3}, \frac{-2}{3})(\frac{1}{3}, \frac{2}{3}; 0^6)'$ | $2_1, 2_3$ | 1 + 1 | $L$ | $C_{10}^0, C_{10}^0$ |
| $(10; \frac{1}{3}, \frac{-2}{3})(\frac{1}{3}, \frac{2}{3}; 0^6)'$ | $1_2, [1_1 + 1_3]$ | 1 + 1 | $L$ | $C_{10}^0, C_{10}^0$ |
| $(10; \frac{-2}{3}, \frac{1}{3})(\frac{1}{3}, \frac{2}{3}; 0^6)'$ | $1_1$ | 1 | $L$ | $C_{10}^0$ |
| $(10; \frac{-2}{3}, \frac{1}{3})(\frac{1}{3}, \frac{2}{3}; 0^6)'$ | $1_3$ | 1 | $L$ | $C_{10}^0$ |
| $(10; \frac{1}{3}, \frac{-2}{3})(\frac{-2}{3}, \frac{-2}{3}; 0^6)'$ | 0 | 1 | $L$ | $C_{10}^0$ |
| $P + 2V_+$ | $(N^L)_j$ | $\mathcal{P}_2(f_+)$ | $\chi$ | Labels |
| $(10; \frac{1}{3}, \frac{-1}{6}, \frac{1}{3}; \frac{1}{3}; 16')$ | 0 | 1 | $L$ | $T_{21}^{2_0}$ |
| $(10; \frac{1}{3}, \frac{-1}{6}, \frac{1}{3}; \frac{1}{3}; 16')$ | $1_2, [1_1 + 1_3]$ | 1 + 1 | $L$ | $D_{10}^1, D_{10}^1$ |
| $(10; \frac{1}{3}, \frac{-1}{6}, \frac{1}{3}; \frac{1}{3}; 16')$ | $2_1, 2_3$ | 1 + 1 | $L$ | $C_{10}^0, C_{10}^0$ |
| $P + 2V_-$ | $(N^L)_j$ | $\mathcal{P}_2(f_-)$ | $\chi$ | Labels |
| $(10; \frac{1}{3}, \frac{-1}{6}, \frac{1}{3}; \frac{1}{3}; 16')$ | 0 | 1 | $L$ | $T_{22}^{10}$ |
| $(10; \frac{1}{3}, \frac{-1}{6}, \frac{1}{3}; \frac{1}{3}; 16')$ | $2_1, 2_3$ | 1 + 1 | $L$ | $D_{10}^1, D_{10}^1$ |
| $(10; \frac{1}{3}, \frac{-1}{6}, \frac{1}{3}; \frac{1}{3}; 16')$ | $1_2, [1_1 + 1_3]$ | 1 + 1 | $L$ | $C_{10}^0, C_{10}^0$ |
| $(10; \frac{1}{3}, \frac{-1}{6}, \frac{1}{3}; \frac{1}{3}; 16')$ | 0 | 1 | $L$ | $C_{10}^0$ |
\[ P = (-1 -1 -1 -1 0; -1 -1 0) (0^i)' \] satisfies the masslessness condition. The \( SU(5) \) representation is \( \bar{5} \). Then, we have
\[
P + 2V = \left( \begin{array}{rrrrr} -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{array} \right) \left( \begin{array}{rrrrr} 1 & 1 & 1 & 1 & 0 \\ \frac{5}{3} & 2 & 0 \\ \frac{7}{3} & 2 & 0 \\ \frac{1}{3} & 2 & 0 \\ 2 & 2 & 0 \end{array} \right)'
\]
which gives \( Q_X = 3 \). From the previous discussion on the right mover condition, we obtain the chirality \( 2\Theta \).

For the \( T_2^\pm \) sectors, only neutral fields under \( SU(5) \times U(1)_X \) arise, which are tabulated in Table 7. Thus, from \( T_2 \) we obtain the following \( SU(5) \times U(1)_X \) representations
\[
\{5_3, 1_{-5}\} + 12 \text{ neutral singlets.}
\]

The \( CTP \) conjugates are provided from \( T_{10} \).

There are four \( SU(2)' \) doublets and one \( 1\bar{6}' \) and one \( 10' \) of \( SO(10)' \).

### 4.4. Twisted sector \( T_3 \)

The shifted momenta in the \( T_3 \) sector must satisfy \( (P + 3V) \cdot a_3 = 0 \mod Z \), viz. Eq. (46). It turns out that there is no massless states satisfying this condition.

### 4.5. Twisted sector \( T_4 \)

The massless condition for the right movers in the \( T_4 \) sector is \( (s + 4\phi_s)^2 = \frac{1}{4} \). Taking a shifted \( 4\phi_s \) as \( \left( \frac{2}{3} \right) \( \frac{1}{3} \) \( \frac{1}{3} \) \( \frac{1}{3} \) \), only the right-handed state \( s = (\Theta---) \) satisfies this condition. So it is left-handed. Now, \( \bar{s} + 4\phi_s \cdot \phi_s = 0 \).

Since \( -\frac{1}{2} (V^2 - \phi^2) = \text{integer} \) for \( k = 4 \), it does not contribute to the phase. So, \( \Delta \) of Eq. (24) is given by \( \Theta_f = (P + 4V) \cdot (\pm) \cdot (\pm) + \sum_j (N^L)j (\hat{\phi})j \). The \( T_4 \) sector is again distinguished by Wilson lines: \( V_0 = V, V_+ = V + a_3 \) and \( V_- = V - a_3 \). These sectors are denoted as \( T_4^0, T_4^+ \) and \( T_4^- \).

\( \tilde{\chi}(\theta^4, \theta^3) \) of the \( T_3 \) sector are \( (27, 3, 3, 3)^3 \), hence the multiplicity is
\[
\mathcal{P}_4 = \frac{3}{12} (1 + \Delta^4 + \Delta^8) (8 + [1 + \Delta + \Delta^2 + \Delta^3]).
\]

So, \( \Delta = e^{2\pi i/12}, e^{2\pi i/6} \) give \( \mathcal{P}_4 = 0 \). \( \mathcal{P}_4 = 9, 6, 6 \) for \( \Delta = 1, -1, e^{2\pi i/4} \). Considering Wilson lines, the nonvanishing multiplicities are \( 3, 2, 2 \) in each \( T_4 \). The massless fields of \( T_4^0 \) are listed in Table 8.

Consider first two rows of Table 8. They are left-handed. The massless condition for the left movers is \( (P + 4V)^2 = \frac{4}{3} \). Since
\[
4V = \left( \begin{array}{rrrrr} 1 & 1 & 1 & 1 & 0 \\ \frac{5}{3} & 2 & 0 \\ \frac{7}{3} & 2 & 0 \\ \frac{1}{3} & 2 & 0 \\ 2 & 2 & 0 \end{array} \right)'
\]
the state
\[
(0, [-1]^4; -2, -2, 0)(-1, -1; 0^6)'
\]
satisfies the masslessness condition. It is \( \bar{5} \) and \( Q_X = -2 \) since \( P + 4V = (1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0) \).

Since \( (P + 4V) \cdot V = 0 \), we obtain \( \Delta = e^{2\pi i 0} \) from (23) and the multiplicity is \( 3 \). The state \( \bar{5} \)
Table 8
Chiral matter fields in the $T_{4}^{0}$, $T_{4}^{+}$, and $T_{4}^{-}$ sectors. There are twelve $SU(2)$ doublets of $d_{1,2,3}^{\pm}$.

<table>
<thead>
<tr>
<th>$P + 4V$</th>
<th>$(N^L)_j$</th>
<th>$\Theta_i$</th>
<th>$\mathcal{P}_4(f_i)$</th>
<th>$\chi$</th>
<th>Labels</th>
</tr>
</thead>
<tbody>
<tr>
<td>(5, -2)</td>
<td>$\frac{1}{2}$, $0^2$</td>
<td>$\frac{1}{2}$, $-\frac{1}{3}$, $0^6$</td>
<td>0</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>(5, -2)</td>
<td>$\frac{1}{2}$, $0^2$</td>
<td>$\frac{1}{2}$, $-\frac{1}{3}$, $0^6$</td>
<td>0</td>
<td>0</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{2}{3}$, $\frac{1}{3}$, $0^6$</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{2}{3}$, $\frac{1}{3}$, $0^6$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{2}{3}$, $\frac{1}{3}$, $0^6$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$P + 4V_+$</td>
<td>$(N^L)_j$</td>
<td>$\Theta_+$</td>
<td>$\mathcal{P}<em>4(f</em>+)$</td>
<td>$\chi$</td>
<td>Labels</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{1}{3}$, $\frac{1}{3}$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{1}{3}$, $\frac{1}{3}$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{1}{3}$, $\frac{1}{3}$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{1}{3}$, $\frac{1}{3}$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$P + 4V_-$</td>
<td>$(N^L)_j$</td>
<td>$\Theta_-$</td>
<td>$\mathcal{P}<em>4(f</em>-)$</td>
<td>$\chi$</td>
<td>Labels</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{1}{3}$, $\frac{1}{3}$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{1}{3}$, $\frac{1}{3}$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{1}{3}$, $\frac{1}{3}$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>(10)</td>
<td>$\frac{2}{3}$, $0^2$</td>
<td>$\frac{1}{3}$, $\frac{1}{3}$</td>
<td>1</td>
<td>2</td>
<td>$\frac{1}{4}$</td>
</tr>
</tbody>
</table>

with $Q_X = 2$.

$$(-2, [-1]^4; -2, -2, 0)(-1, -1; 0^6)'$$

also satisfies the masslessness condition. Since $(P + 4V) \cdot V = -\frac{1}{2}$, we obtain $\Delta = e^{2\pi i (-\frac{1}{2})}$ from (23) and the multiplicity is 2.

In the $T_{4}^{\pm}$ sectors, the phases in Eq. (26) are respectively given by

$$\Theta_+ = (P + 4V_+) \cdot V_+ + \sum_j (N^L)_j \hat{\phi}_j + \frac{1}{3},$$

(70)

$$\Theta_- = (P + 4V_-) \cdot V_- + \sum_j (N^L)_j \hat{\phi}_j,$$

(71)

where “$\frac{1}{3}$” in Eq. (70) comes from “$-\frac{4}{2}(V_+^2 - \phi_s^2)$”.
It turns out that from the $4(V \pm a_3)$ sectors, only neutral fields under $SU(5) \times U(1)_X$ arise, which are listed in Table 8.

Thus, the massless states in $T_4(+T_8)$ sectors are

$$5_{-2} + 2(5_{-2}, \tilde{5}_{2}) + 30 \text{ neutral singlets} + CTP\text{ conjugates.} \quad (72)$$

There are twelve $SU(2)'$ doublets.

4.6. Twisted sector $T_5$

In the $T_5$ (and $T_5^\pm$) sector of the $Z_{12-1}$ orbifold, only the right-handed chirality states appear as massless states from the right mover condition.

The massless condition for the right mover in the $T_5$ sector is $(s + 5\phi_s)^2 = \frac{11}{24}$. Taking a shifted $5\phi_s$ as $(\frac{1}{12} - \frac{4}{12} \, 5 \frac{5}{12})$, only the right mover $s = (\oplus-+-)$ satisfies this condition. So it is right-handed. Now, $(5 + 5\phi_s) \cdot \phi_s = -\frac{1}{8}$.

Note that $-\frac{k}{2}(V^2 - \phi^2) = -\frac{1}{4}$ for $k = 5$. So, the phase in Eq. (24) is given by $\Theta_0 = (P + 5V) \cdot V + \sum_j (N^L_j)(\hat{\phi})_j - \frac{1}{8}$. The $T_5$ sector is again distinguished by Wilson lines: $V_0 = V, V_+ = V + a_3$, and $V_- = V - a_3$. These sectors are denoted as $T_5^0, T_5^+, T_5^-$. In the $T_5$ sectors, only the states with $\Theta_0, \pm = 0$ survive the GSO projection Eqs. (23), (25) since all $\tilde{\chi}(5,l)$s are the same.

Only neutral and vector-like pairs of $SU(5)$ singlets arise in $T_5$. In the $T_5^+$ sector, only the states with $(P + 5V_+) \cdot V_+ + \sum_j (N^L_j)(\hat{\phi})_j = -\frac{1}{24}$ survive the GSO projection Eq. (23). In the $T_5^-$ sector, only the states with $(P + 5V_-) \cdot V_- + \sum_j (N^L_j)(\hat{\phi})_j = -\frac{3}{8}$ survive. These are shown in Table 9.

The $CTP$ conjugates of $T_5$ appear in $T_7$.

5. Summary of matter spectra

Collecting all the flipped $SU(5)$ model fields, we obtain the following:

$$U: \quad (1_{-5} + 5_3 + 10_{-1})_{U_3}, \quad (\tilde{5}_2)_{U_2}, \quad (1_{-5} + 5_3 + 10_{-1})_{U_1}, \quad (1_0)_{U_2}, \quad (1_0)_{U_1},$$

from the untwisted sector, and

$$T_6: \quad \overline{10}^L_{-1} \pm \{ 2(1_5 + 1_{-5} + \tilde{5}_{-3} + 5_3) + 3(10_1 + 10_{-1}) \}^L + 22\{10\}, \quad (74)$$

$$T_2: \quad 1^L_{-5} + 5^L_3 + 12\{10\}, \quad (75)$$

$$T_4: \quad 5^L_{-2} + 2(5_{-2} + \tilde{5}_2)^L + 30\{10\}, \quad (76)$$

from the twisted sectors. The chiral matter resulting from this spectra constitutes three families of quarks and leptons.

In addition, we obtain G-exotics and E-exotics from $T_1$ and $T_5$ sectors,

$$T_1: \quad 2(\tilde{5}_{1/2})^L + 2(5_{+1/2})^L + 7(1_{-5/2})^L + 7(1_{+5/2})^L, \quad (77)$$

$$T_5: \quad 2(5_{+1/2})^R + 2(\tilde{5}_{-1/2})^R + 7(1_{+5/2})^R + 7(1_{-5/2})^R. \quad (78)$$

From (2), we note that G-exotics carry $Q_{em} = \pm \frac{1}{6}$ quarks and $Q_{em} = \pm \frac{1}{2}$ E-exotics and E-exotics with $X = \pm \frac{5}{2}$ have $Q_{em} = \pm \frac{1}{2}$. All these exotics can be removed if all $U(1)s$ except the
Table 9
Chiral matter fields satisfying $\Theta_{0,\pm} = 0$ in the $T^0_3$ and $T^\pm_3$ sectors. There are two $SU(2)'$ doublets with $(\frac{5}{6}, \frac{1}{6})'$.

<table>
<thead>
<tr>
<th>$P + 5V$</th>
<th>$(N^L)_j$</th>
<th>$\mathcal{P}_3(f_0)$</th>
<th>$\chi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5_{1/2}': \frac{5}{12}, 0, \frac{6}{12})(\frac{1}{6}, -\frac{1}{6}; 06)'$</td>
<td>0</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(5_{1/2}': \frac{5}{12}, 0, \frac{6}{12})(\frac{1}{6}, -\frac{1}{6}; 06)'$</td>
<td>1</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(5_{-1/2}: \frac{1}{12}, \frac{6}{12}, 0)(\frac{1}{6}, -\frac{1}{6}; 06)'$</td>
<td>2</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(5_{1/2}: \frac{7}{12}, 0, \frac{6}{12})(\frac{1}{6}, -\frac{1}{6}; 06)'$</td>
<td>3</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(5_{1/2}: \frac{5}{12}, 0, \frac{6}{12})(\frac{1}{6}, -\frac{1}{6}; 06)'$</td>
<td>1</td>
<td>1+1</td>
<td>R</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{1}{12}, \frac{5}{12}, 0)(\frac{1}{6}, -\frac{1}{6}; 06)'$</td>
<td>0</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{1}{12}, \frac{5}{12}, 0)(\frac{1}{6}, -\frac{1}{6}; 06)'$</td>
<td>13, 51, ${11 + 13}$</td>
<td>1+1+1</td>
<td>R</td>
</tr>
<tr>
<td>$P + 5V_+$</td>
<td>$(N^L)_j$</td>
<td>$\mathcal{P}<em>5(f</em>+)$</td>
<td>$\chi$</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{5}{12}, \frac{4}{12}, \frac{2}{12})(\frac{5}{6}, \frac{1}{6}, \frac{1}{3}; 05)'$</td>
<td>0</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{5}{12}, \frac{4}{12}, \frac{2}{12})(\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}; 05)'$</td>
<td>2</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{1}{12}, \frac{5}{12}, \frac{2}{12})(\frac{5}{6}, \frac{1}{6}, \frac{1}{3}; 05)'$</td>
<td>1</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{1}{12}, \frac{5}{12}, \frac{2}{12})(\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}; 05)'$</td>
<td>3</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$P + 5V_-$</td>
<td>$(N^L)_j$</td>
<td>$\mathcal{P}<em>5(f</em>-)$</td>
<td>$\chi$</td>
</tr>
<tr>
<td>$(5_{-1/2}: \frac{1}{12}, \frac{2}{12}, \frac{4}{12})(\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}; 05)'$</td>
<td>0</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{5}{12}, \frac{8}{12}, \frac{2}{12})(\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}; 05)'$</td>
<td>0</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(1_{5/2}: \frac{5}{12}, \frac{8}{12}, \frac{2}{12})(\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}; 05)'$</td>
<td>1</td>
<td>1</td>
<td>R</td>
</tr>
<tr>
<td>$(1_{-5/2}: \frac{1}{12}, \frac{2}{12}, \frac{8}{12})(\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}; 05)'$</td>
<td>1</td>
<td>1</td>
<td>R</td>
</tr>
</tbody>
</table>

$U(1)_Y$ are broken at the GUT scale and a sufficient number of neutral singlets develop VEVs, and hence there are not left with dangerous half-integer charged fields below the GUT scale. It will be explained in Section 6.1. The lightest of these half-integer charged fields (LHIC) is absolutely stable since all the light observable SM fields (including color singlet composites) are integer charged. If the mass of LHIC is much larger than the reheating temperature after inflation, we expect that most of LHIC are diluted away by inflation.

For the hidden sector, there appear 20 $SU(2)'$ doublets and one $\overline{10}'$ and one $10'$ of the hidden $SO(10)'$.

6. Yukawa couplings

Nonvanishing couplings of vertex operators are constructed by satisfying the $Z_{12-1}$ symmetry [29,35,37]. It is summarized in [7]. In our notation for the shift vector, basically it amounts for the operator $O_AO_BO_C \cdots$ to satisfy

\begin{align}
\text{Invariance under the group space shift } V, \quad \text{and} \\
\text{Invariance under the internal space shift } \phi_s.
\end{align}

For the shift $V$, it is easy to check the modular invariance: The relevant vertex operators have only to satisfy the gauge invariance. The invariance under the shift $\phi_s$ belongs to a generalized Lorentz shift and the condition is sometimes called the $H$-momentum conservation. The (bosonic) $H$-
momentum is defined as

$$R_i = (\tilde{s} + k\phi_s + \bar{r})_i - (N_i^{L} - N_i^{R}), \quad i = 1, 2, 3,$$

(81)

where $\bar{r}$ is $(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2})$ for left-handed states ($\equiv \bar{r}_-$) and $(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$ for right-handed states ($\equiv \bar{r}_+$). As discussed earlier, $\tilde{s}$ should satisfy the mass-shell condition, $|(\frac{1}{2} \chi, \tilde{s}) + k\phi_s|^2 = 2c_k$. $R_i$ can be interpreted as a discrete $R$-charge. Thus, neglecting oscillator numbers, the $H$-momenta for $\mathbf{Z}_{12^\pm1}$ twist are

$$U_1: (\begin{array}{c} -1 \ 0 \ 0 \end{array}), \quad U_2: (\begin{array}{c} 0 \ 1 \ 0 \end{array}), \quad U_3: (\begin{array}{c} 0 \ 0 \ 1 \end{array}),$$

$$T_1: (\begin{array}{c} -\frac{7}{12} \ \frac{4}{6} \ \frac{1}{12} \end{array}), \quad T_2: (\begin{array}{c} -\frac{1}{6} \ \frac{4}{6} \ \frac{1}{6} \end{array}), \quad T_3: (\begin{array}{c} -\frac{3}{4} \ 0 \ \frac{1}{4} \end{array}),$$

$$T_4: (\begin{array}{c} -\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \end{array}), \quad T_5: (\begin{array}{c} \frac{1}{12} \ -\frac{4}{12} \ -\frac{7}{12} \end{array}), \quad T_6: (\begin{array}{c} -\frac{1}{2} \ 0 \ \frac{1}{2} \end{array}),$$

(82)

which are used to check the generalized Lorentz invariance.

As an example, consider the $T_6$ $H$-momentum, $(\frac{-1}{2} 0 \ \frac{1}{2})$. It is derived in the following way. The right mover mass-shell condition is

$$\frac{1}{2} M_R^2 = (s + 6\phi_s)^2 = 2c = \frac{1}{2}$$

(83)

where $6\phi_s = (\frac{30}{12} \ \frac{24}{12} \ \frac{6}{12})$. There are two solutions of Eq. (83),

$$s_+ = \left(\Theta; \ \frac{-5}{2}, \ -\frac{3}{2}, \ -\frac{1}{2}\right) \quad \text{and} \quad s_- = \left(\Theta; \ \frac{-5}{2}, \ -\frac{5}{2}, \ -\frac{1}{2}\right).$$

For the left-handed states appearing in Table 4, let us focus on $s_-$ whose corresponding sum-to-odd-integer solution is $(-3 -2 0)$. So bosonization of $\tilde{s}_-$ is $\tilde{s}_- + \bar{r}_-$. Thus, $H$-momentum, in analogy with $P + kV$, is $(\tilde{s}_- + \bar{r}_-) + k\phi_s = (\frac{-1}{2} 0 \ \frac{1}{2})$ appearing in Eq. (82).

The $H$-momenta conservation conditions with $\phi_s = (\frac{5}{12}, \ \frac{4}{12}, \ \frac{1}{12})$ can be reformulated only in terms of the bosonic $H$-momenta as follows:

$$\sum_z R_1(z) = -1 \ \text{mod} \ 12, \quad \sum_z R_2(z) = 1 \ \text{mod} \ 3, \quad \sum_z R_3(z) = 1 \ \text{mod} \ 12,$$

(84)

where $z (\equiv A, B, C, \ldots)$ denotes the index of states participating in a vertex operator. In addition, space group selection rules requires a vertex operator with $z$-states in $T^{m,f}_k$ sector ($k = 0$ for the untwisted sector) should satisfy

$$\sum_z k(z) = 0 \ \text{mod} \ 12,$$

(85)

$$\sum_z [km_f](z) = 0 \ \text{mod} \ 3.$$  

(86)

One can easily check the following cubic couplings are allowed in $\mathbf{Z}_{12^{-1}}$ orbifold models fulfilling Eqs. (84) and (85), if $N_i^{L} = N_i^{R} = 0$ [35,37]:

$$U_1 U_2 U_3, \quad T_6 T_6 U_2, \quad T_4 T_4 T_4, \quad T_2 T_4 T_6, \quad T_1 T_4 T_7.$$  

(87)

Note that in considering the superpotential couplings, one should consider only the same chirality, and in our model there is no massless states from the $T_3$ and $T_9$ sectors.
Table 10

$H$-momenta for some combinations of neutral singlets under $SU(5) \times U(1)_X$ appearing in our model. All the combinations are neutral under all gauge symmetries in this model, and fulfill the space group selection rules.

<table>
<thead>
<tr>
<th>Comb. of singlets</th>
<th>$H$-momenta</th>
<th>Comb. of singlets</th>
<th>$H$-momenta</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_1 \bar{h}_1$</td>
<td>$(-1 \ 0 \ -1)$</td>
<td>$h_1 \bar{h}_2$</td>
<td>$(-1 \ 0 \ 1)$</td>
</tr>
<tr>
<td>$h_2 \bar{h}_2$</td>
<td>$(-1 \ 0 \ 3)$</td>
<td>$h_1 \bar{h}_4$</td>
<td>$(-2 \ 0 \ 0)$</td>
</tr>
<tr>
<td>$h_3 \bar{h}_3$</td>
<td>$(1 \ 0 \ 1)$</td>
<td>$h_2 \bar{h}_3$</td>
<td>$(0 \ 0 \ 2)$</td>
</tr>
<tr>
<td>$h_4 \bar{h}_4$</td>
<td>$(-3 \ 0 \ 1)$</td>
<td>$h_2 \bar{h}_4$</td>
<td>$(-2 \ 0 \ 2)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Comb. of doublets</th>
<th>$H$-momenta</th>
<th>Comb. of doublets</th>
<th>$H$-momenta</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1^+ d_2^+ \bar{h}_1$</td>
<td>$(-1 \ 3 \ 0)$</td>
<td>$D_1^+ d_2^+ \bar{h}_3$</td>
<td>$(0 \ 3 \ 1)$</td>
</tr>
<tr>
<td>$D_1^+ d_2^+ \bar{h}_2$</td>
<td>$(-1 \ 3 \ 2)$</td>
<td>$D_1^+ d_2^+ \bar{h}_4$</td>
<td>$(-2 \ 3 \ 1)$</td>
</tr>
</tbody>
</table>

For future convenience, we display the $H$-momenta for combinations of some singlets appearing in our model in Table 10.

In Ref. [30], it has been shown that one can always find a vacuum where only neutral singlets under a symmetry of interest develop VEVs of string scale preserving the F and D flatness conditions. This is possible because (a) once a superpotential term $W$ is allowed by the selection rules, then $W^{N+1}$ is also allowed in the $Z_N$ orbifold model, and (b) there exists in general a transformation rescaling the VEVs leaving intact the above F flatness conditions but making accordingly D flatness conditions satisfied by adjusting VEVs. In our model, there are enough superpotential terms constructed with only neutral singlets, e.g. $W = W_S + W_{13}^2 + W_{25}^2 + \cdots$, where

$$W_S = (C_1^+ s_2^+ \bar{h}_1)(C_1^- s_2^- h_4) + (C_1^+ s_2^+ \bar{h}_4)(C_1^- s_2^- h_1) + (C_1^- s_2^- h_2)(C_2^- s_2^- h_1) + (C_1^- s_2^- h_1)(C_2^- s_2^- h_2) + \cdots$$

where the string scale is set $M_{\text{string}} = 1$. We assume a vacuum where only neutral singlets develop large string scale VEVs.

6.1. Flipped SU(5) spectrum

There exist $(10_1, \bar{10}_{-1})$ whose VEV (in the $SU(2)$ singlet direction $v^c$) breaks the flipped SU(5) down to the standard model. Also, there exist the needed electroweak Higgs fields $(\bar{5}_2, 5_{-2})$.

In $T_1$ and $T_3$, there appear G-exotics and E-exotics. These are removed by $T_1 T_4 T_7$ couplings of Eq. (87) via $(1_0(T_4))$ and other singlet’s VEVs. For instance, $(1_{5/2}; \frac{4}{12}, 0, \frac{2}{12})(\frac{1}{6}, \frac{1}{6}, 0)^c$ with $N_j = 1$ in $T_1^0$, and $(1_{-5/2}; \frac{7}{12}, 4, \frac{2}{12})(\frac{1}{6}, \frac{1}{6}, -\frac{1}{2}, 0)^c$ with $N^L_j = 1$ in $T_7^0$ get a mass from
the coupling with \( \langle s_1^+ \rangle \) in \( T_4^+ \). Similarly \((1_-, s_2^\pm; -\frac{7}{12}, -\frac{1}{12}, -\frac{1}{12}), (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}) \)' with \( N_j^L = 13 \) in \( T_4^- \),
and \((1\uparrow / 2; -\frac{1}{12}, \frac{1}{12}, 0, 0, 0 \rangle \)' with \( N_j^L = 13 \) in \( T_4^0 \) achieve a mass also from \( s_1^+ \). It turns out that the contributions of oscillator numbers carried by the other states to the \( H \)-momenta can be always cancelled by (multi-) products of \( C_1 \) singlets with \( C_2 \) light fields, \( C_3 \) doublets, \( C_4 \) triplets, \( C_5 \) \( \bar{h}_3 \), \( C_6 \) \( s_2 \bar{h}_1 \), etc. When \( N_j^L = \{12 + 13\} \) is involved in a \( T_1 T_7 T_4 \) vertex operator, for instance, one could additionally multiply \( C_1^+ s_2^+ \bar{h}_4 \) to the vertex operator in order to cancel the oscillator number contribution. When \( N_j^L = 43 \) is involved, e.g. \( C_1^- s_2^- h_3^4 \) needs to be multiplied.

In \( T_6 \) and \( T_4 \), there appear vector-like representations \((101 + 10\linebreak1 - 1) \), \( (1\uparrow / 2) \), and \( (5\downarrow - 5\downarrow / 2) \). These are removed by the survival hypothesis that all vector-like representations are removed at the GUT scale [36]. Indeed, it is so in our model by allowing large VEVs to neutral singlets. Thus, we obtain just only one \( 5\downarrow / 2 \) from \( T_4 \) and \( 5\downarrow / 2 \) from the untwisted sector. One can anticipate that this may happen if the twisted sectors provide large Yukawa couplings. Indeed, such couplings are present.

The fields \( 2(5\downarrow - 5\downarrow / 2) \) of \( T_4 \) can be removed by \( T_4 T_4 T_4 \) couplings where one \( T_4 \) is \( s_0^L \) in Table 8. The vector-like representations in \( T_6 \) are also removed by the couplings, e.g. with \((C_1^+ s_2^+ \bar{h}_1)(C_1^- s_2^- h_1) \), \((C_2^+ s_2^+ \bar{h}_3)(C_1^- s_2^- h_3) \), etc. Thus, \( 2(1\uparrow / 2) \) from \( T_4 \) and \( 5\downarrow / 2 \) from the untwisted sector. One can develop GUT scale VEVs. Then, there result the following light fields,

\[
U: \quad (1\uparrow / 2 + 5\downarrow + 101)^R_{U_1} + (1\uparrow / 2 + 5\downarrow + 101)^R_{U_2} + (5\downarrow - 5\downarrow / 2)^R_{U_3}, \quad (88)
\]

\[
T_6: \quad 10^R, \quad (89)
\]

\[
T_4^0: \quad 1\downarrow + 5\downarrow, \quad (90)
\]

\[
T_4^0: \quad 5\downarrow / 2. \quad (91)
\]

These constitute the three families and one pair of Higgs quintets. Relabeling \( R \) to \( L \), we can obtain the standard form of left-handed \( W^\pm \) interactions. In the remainder, however, we will keep \( R \) to compare with the entries of tables.

It is interesting to note that the pair of Higgs quintets, \( (5\downarrow - 5\downarrow / 2) \), survives the above analysis. Certainly, it is not allowed to write \( \text{M}_{\text{GUT}} \cdot 5\downarrow - 5\downarrow / 2 \) since there is no coupling of the form \( U_2 T_4 \). In fact, it is not easy to construct a \( 5\downarrow - 5\downarrow / 2 \) consistent with the \( H \)-momentum conservation. Thus Higgs doublet mass can be far below the GUT scale, which is a good thing. But the colored scalars in \( (5\downarrow - 5\downarrow / 2) \) must be removed at high energy scale toward MSSM. It is achieved by the doublet–triplet splitting mechanism we discuss below.

### 6.2. GUT breaking

The GUT breaking in the flipped \( SU(5) \) model proceeds by \( \langle 101 \rangle = \langle 10\linebreak1 - 1 \rangle = \text{M}_{\text{GUT}} \), which is a D-flat direction. This vector-like representation, \( 101 + 10\linebreak1 - 1 \), is present in the \( T_6 \) sector. Note that the \( H \)-momentum of \( \langle 101, 10\linebreak1 - 1 \rangle \) is \((-1, 0, 1, 0)\).

The \( T_6 \) sector also contains twenty two \( Q_{em} = 0 \) singlets. Some combinations of them, e.g. \( h_1 \bar{h}_2 \) in Table 10 also give the \( H \)-momentum of \( (-1, 0, 1, 0) \). \( h_3 \bar{h}_4 \), \( h_2 \bar{h}_1 \), \( h_4 \bar{h}_3 \) also provide the same \( H \)-momentum. The GUT scale VEV of them could induce \( \langle 101 \rangle = \langle 10\linebreak1 - 1 \rangle = \text{M}_{\text{GUT}} \) along an F-flat direction, for instance, through a non-renormalizable superpotential,

\[
W = (D_1^+ d_2^+ \bar{h}_1)(C_1^+ s_2^+ \bar{h}_1)[10^H \bar{10}^H_{-1} - h_1 \bar{h}_2 - h_3 \bar{h}_4 + \ldots] \quad (92)
\]
with $\langle D_1^+ \rangle$ or $\langle d_2^+ \rangle$ vanishing. Note that there are 20 $SU(2)'$ doublets. Thus, the gauge coupling of $SU(2)'$ would not blow up at lower energies. The neutrino direction of $\langle 10_1 \rangle$, $\langle \overline{10}_{-1} \rangle$ allows the symmetry breaking

$$SU(5) \times U(1)_X \rightarrow SU(3)_c \times SU(2) \times U(1)_Y.$$ (93)

6.3. Doublet–triplet splitting

Let us call the three vector-like ten and tenbars of Eq. (74) as Higgs fields, $10^H_1$ and $\overline{10}^H_{-1}$. Among the three two are purely vector-like and removed at the GUT scale. One remaining vector-like pair joins the Higgs mechanism. Let us consider this remaining pair.

The Higgs $10^H_1$ and $\overline{10}^H_{-1}$ contain $\{q, d^c, \nu^c\} + \{\bar{q}, \bar{d}^c, \bar{\nu}^c\}$ in terms of the standard model quantum numbers. $\{\nu^c, \bar{\nu}^c\}$ obtain GUT scale masses from Eq. (92) when $SU(5) \times U(1)_X$ is broken. $\{q, \bar{q}\}$ contained in $10^H_1$ and $\overline{10}^H_{-1}$ are absorbed by the heavy gauge sector. On the other hand, $\{d^c, \bar{d}^c\}$ still remain light. In order to make the standard model vacuum stabilized, somehow they should get superheavy masses.

Let us consider $W = 10^H_1 10^H_1 \tilde{5}_h + \overline{10}^H_{-1} \overline{10}^H_{-1} \tilde{5}_h$, where $\tilde{5}_h$ and $\tilde{5}_h$ indicate the five-plets inducing electroweak symmetry breaking [38,39]. When $10^H_1$ and $\overline{10}^H_{-1}$ develop VEVs in the $\langle \nu^c \rangle = \langle \bar{\nu}^c \rangle = M_{GUT}$ direction, $\{d^c, \bar{d}^c\}$ in $10^H_1$, $\overline{10}^H_{-1}$ and triplets $(D, \bar{D})$ included in $\tilde{5}_h, \tilde{5}_h$ pair up to be superheavy [40]. This achieves the doublet–triplet splitting in flipped $SU(5)$.

This mechanism can be realized also in our model. The $[\overline{10}^H_{-1} 10^H_1 \tilde{5}_h]^L$ term arises from $T_6 T_2 U_2$, which satisfies the $H$-momentum conservation. The $[10^H_1 10^H_1 \tilde{5}_h]^L$ is still also possible from highly non-renormalizable interactions, e.g. $[10^H_1 10^H_1 \tilde{5}_h]^L (s_{32} s_2^0)(C^*_2 s_2^0 h_3)(C^*_2 s_2^0 h_3)$, which satisfies all selection rules discussed above. Thus we should assume cutoff scale VEVs for the neutral singlets.

Thus, we obtain the so-called MSSM spectra with one pair of Higgs doublets at cubic level. However, the Higgs doublets would obtain mass since there exist a lot of singlets. Indeed, there exist many couplings. The simplest two terms for $\mu$ are

$$\left[(\tilde{5}_2; -1, 0, 0)(0^8)\right]_{U_2} \left[(\tilde{5}_{-2}; \frac{-1}{3}, 0, 0)\left(\frac{-1}{3}, \frac{-1}{3}; 0^0\right)\right]_{T_4^0} \left[s_2^{0} (s_2^{0})^2 + f_\mu s_2^{0} s_4^{0}\right]_{T_4^0}$$ (94)

where $f_\mu$ is a relative strength. Before, $s_2^{0}$ was needed for charged lepton masses. On the other hand, $s_2^{0}$ was needed for mixing of charged leptons. Let us set $s_2^{0}$ a free parameter. Assuming that $\tilde{5}_2$ and $\tilde{5}_{-2}$ obtain VEVs, the F-flat direction chooses $s_2^{0} = -f_\mu s_4^{0}/2s^0$ so that the $\mu$ term is of order $-f_\mu^2 s_4^{02}/4s^0$. This can be linked to the charged lepton masses. For example, if we choose $s_4^{0} \sim 10^{-5}$ from electron mass and $s^0 \sim O(1)$, we require $f_\mu = O(10^{-2})$ to obtain a TeV scale $\mu$ term. This fine tuning is a huge improvement over a fine tuning of $10^{-15}$. Since we neglected many higher order terms, this is just an illustration of smoothing the fine tuning problem. The $s_2^{0}$ mimics the axion multiplet of the $\mu$ solution [41].

6.4. Fermion masses

- There is one cubic coupling relevant for $u$-type quarks and Dirac neutrinos, $T_6 T_2 T_4$, which is interpreted as the top quark and (Dirac) tau neutrino Yukawa coupling,

$$t \text{ quark } + \text{(Dirac) } \tau \text{ neutrino coupling: } \left[10^R_1 \tilde{5}_{-3} \tilde{5}_2\right]^R.$$ (95)
Non-renormalizable terms allow the other $u$-type quark (Dirac neutrino) Yukawa couplings. Most of all, through the couplings with $C^0_5$, $C^0_6$ in $T^0_3$ and $h_1$, $\bar{h}_3$ in $T_6$, $[\bar{T}^0_{-1}(T_6)\Phi_3(U_1)\Phi_{-2}(T^0_4)]^L$, $[\bar{T}^0_{-1}(T_6)\Phi_3(U_3)\Phi_{-2}(T^0_4)]^L$, $[\bar{T}^0_{-1}(U_3)\Phi_3(T^0_4)\Phi_{-2}(T^0_4)]^L$, and $[\bar{T}^0_{-1}(U_1)\Phi_3(T^0_2)\Phi_{-2}(T^0_4)]^L$ are allowed. With $h_1 C^0_5$, $\bar{h}_3 C^0_6$, $\bar{h}_3 C^0_5$, and $h_1 C^0_6$, the masses and mixing for the first two families of $u$-type quarks and Dirac neutrinos are possible.

- The bottom quark mass arises in terms of $U_2$ Higgs doublet. Indeed, such a coupling is present from $T_6T_6U_2$. If the coupling strength of $T_6T_2T_4$ and $T_6T_6U_2$ are comparable, a large $\tan \beta$ is needed to obtain $m_t/m_b \sim 35$. But the couplings depend on the location of the respective fields and one may treat the ratio as a free parameter of order 1 [42].

In fact $[\Phi_0(T_1)\Phi_0(T_3)\Phi_{-2}(U_2)]^R$ exists. For smallness of it we could assume a proper volume of the 6-dimensional compact space. As an alternative way, one could consider (1, 1) and (2, 2) components of the $d$-quark mass matrix. They can be induced with $\langle h_1h_1 \rangle$ and $\langle \bar{h}_3\bar{h}_3 \rangle$. By $\langle h_1 \rangle$ and $\langle \bar{h}_3 \rangle$, the mixing between the first two and the third families of $d$-type quarks are also permitted.

- For the mass of charged lepton in the twisted sector $T^0_3$, we need a coupling containing $[\bar{1}_{-5}(T^0_3)\Phi_3(T^0_3)\Phi_{-2}(U_2)]^L$. This coupling is possible by being supplemented with $(s^0_5)^2$. On the other hand, cubic couplings $[\Phi_0(U_1)\Phi_{-3}(U_3)\Phi_{-2}(U_2)]^R$ and $[\Phi_3(U_3)\Phi_{-3}(U_1)\Phi_{-2}(U_2)]^R$ are possible. It may be that $\tau$ is placed in the untwisted sector. In this sense, the leptonic sector does not go parallel to the quark sector. The $b - \tau$ unification is not achieved in this model. (1, 1) and (2, 2) components of the charged lepton mass matrix are generated via, e.g. $\langle \bar{h}_1h_1 \rangle$ and $\langle \bar{h}_3h_3 \rangle$. The mixing terms between the first two and the third families of charged leptons should be mediated by $\langle \bar{h}_3s^0_2 \rangle$ and $\langle \bar{h}_1s^0_3 \rangle$.

- In flipped $SU(5)$, Majorana neutrino masses arise from $[\langle \Phi_0(T_1) \rangle \cdot \Phi_0(T_3) \cdot \Phi_{-2}(U_2)]^L$. The $H$-momentum of this operator is $(2, 0, 2)$. Thus this operator can be induced when supplemented, e.g. by the coupling with $(C^+_1 s^+_2 h_3)(C^+_2 s^+_2 \bar{h}_3)(C^-_1 s^+_2 \bar{h}_1)(C^-_2 s^+_2 h_3)^2$. The other components of the Majorana neutrino mass matrix should require additional singlet VEVs such as $h_1$, $\bar{h}_3$, $(h_1)^2$, and $(h_3)^2$. If heavy Majorana neutrino masses are around $10^{14}$ GeV, thus we obtain the $\nu_\tau$ mass of order 0.1 eV,

$$m_{\nu_\tau} \simeq \frac{m^2_{\text{top}}}{(10^{14} \text{ GeV})} \sim 0.1 \text{ eV}$$

(96)

for $m_{\text{top}} = m_{\text{Dirac neutrino}}$, which is valid in flipped $SU(5)$.

6.5. $R$-parity

If we consider cubic couplings Eq. (87), we can define an $R$-parity in the standard way, $R = -1$ for matter fermions and $R = +1$ for Higgs bosons.

Firstly, consider the coupling $\Phi_0(U_1)\Phi_0(U_3)\Phi_{-2}(U_2)$. A nontrivial parity can be defined as

$$R = -1 \quad \text{for } \Phi_0(U_1), \quad \Phi_0(U_3), \quad \Phi_{-3}(U_1), \quad \Phi_{-3}(U_3), \quad \Phi_3(U_1), \quad \Phi_3(U_3),$$

(97)

$$R = +1 \quad \text{for } \Phi_{-2}(U_2).$$

(98)

As discussed above, mixing between the first two families of matter in the untwisted sector and the third family of matter in $T_3$ and $T^0_4$ are always possible if VEVs of some neutral singlets are supposed. Such neutral singlets should also be inert under all symmetries relevant at low energies. Otherwise, symmetries must be broken at low energies by their VEVs. Even with the mixing terms between untwisted and twisted matter fields, the $R$-parity relevant in low energies
can still be defined by assigning \( R = 1 \) for the neutral singlets developing VEVs and

\[ R = -1 \] for \( 1_{-1}(T_6), 5_{-1}^L(T_2), 1_{-5}^L(T_2). \]  \hspace{1cm} (99)

Then, the allowed Yukawa coupling \( T_6 T_2 T_4 \) determines

\[ R = +1 \] for \( 5_{-2}(T_4). \]  \hspace{1cm} (100)

Thus, \( R \)-parity can survive down to low energies and hence \( R \)-parity conservation for proton longevity is fulfilled in the present model.

7. Electroweak hypercharge in flipped \( SU(5) \)

7.1. GUT value of weak mixing angle

Flipped \( SU(5) \) was originally considered as a subgroup of \( SO(10) \) [9], which can be called \( SO(10) \)-flipped \( SU(5) \). In this case, the GUT value for \( \sin^2 \theta_W \) should be \( \frac{3}{8} \). In this case, symmetry breaking must proceed via adjoint Higgs field. The shift vector must take a form

\[ V = \begin{cases} 
(0, 0, 0, 0, x, y, z)(\cdots)', \\
\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, x, y, z\right)(\cdots)' 
\end{cases} \]  \hspace{1cm} (101)

so that an \( SO(10) \) group is obtained.

String flipped \( SU(5) \), for example Eq. (28) with five \( \frac{1}{3} \)s, is basically different from \( SO(10) \) flipped \( SU(5) \) even though it includes \( SO(10) \) flipped \( SU(5) \) if the shift vector takes the form Eq. (101). If the electroweak \( SU(2) \times U(1)_Y \) is embedded in a simple group in a field theory GUT, the GUT value of \( \sin^2 \theta_W \) is \( \frac{3}{8} \) as calculated from

\[ \sin^2 \theta_W = \frac{\text{Tr} T_3^2}{\text{Tr} Q_{em}^2}. \]  \hspace{1cm} (102)

Many possibilities of \( \sin^2 \theta_W \) appear because of many possibilities of embedding the electroweak hypercharge \( Y \) in an Abelian group. In the \( E_8 \times E'_8 \) heterotic string, the embedding of \( Y \) in \( U(1)_Y \) is basically looked from the untwisted sector spectrum, which sets the embedding in \( E_8 \). Thus, in string compactification, \( \sin^2 \theta_W \) depends on the spectrum in the untwisted sector [7]. Therefore, if the untwisted sector spectrum includes \( 16_{\text{flip}} \), then \( \sin^2 \theta_W \) is the same as that of \( SO(10) \). On the other hand, if \( 16_{\text{flip}} \) cannot be obtained from the fields in the \( U \) sector, \( \sin^2 \theta_W = \frac{3}{8} \) is not warranted. In the present model, a complete multiplet \( 16_{\text{flip}} \) appears in the \( U \) sector, we obtain \( \sin^2 \theta_W = \frac{3}{8} \). This is explicitly shown below.

The electroweak hypercharge \( Y \) is a combination of \( SU(5) \) and \( U(1)_X \) generators, \( Y = \frac{1}{3} (X + Y_5) \). As will be shown in the next section, the \( U(1)_Y \) gauge coupling is

\[ g_Y^{-2} = \frac{g_5^{-2}}{15} + \frac{g_X^{-2}}{25u^2}, \]  \hspace{1cm} (103)

where \( u \) denotes an employed unit of \( U(1)_X \) charges. So far we tacitly supposed \( u = 1 \), but its absolute value should be determined by the string theory with \( g_5 = g_X \) at the compactification scale. Along the standard model direction, Eq. (103) is still valid above the flipped \( SU(5) \) breaking scale. Thus at the compactification scale, where \( SU(2) \) gauge coupling \( g_2 \) is identified
with $g_5$, the Weinberg angle is given by
\[
\sin^2 \theta_W^0 = \frac{1}{1 + \frac{g_5^2}{g_X^2}} = \frac{1}{1 + \left(\frac{1}{15} + \frac{1}{25u^2}\right)}.
\] (104)

Including the unit factor ‘$u$’, the $U(1)_X$ charge operator, $Q^A_X \equiv -2(1, 1, 1, 1, 0, 0, 0) \times (0)^5' \times u$ can be expressed in terms of a proper orthonormal basis vector $q_5$ [43],
\[
Q^A_X = \frac{1}{\sqrt{2}} c_5 q_5,
\] (105)
where
\[
c_5 \equiv -2\sqrt{10}u \quad \text{and} \quad q_5 \equiv \frac{1}{\sqrt{5}} (1, 1, 1, 1, 0, 0, 0)(0)^5).
\] (106)

With this expression, $g_5^2/g_X^2$ can be calculated in the string theory framework [7],
\[
\frac{g_5^2}{g_X^2} = c_5^2 = 40u^2.
\] (107)

Hence, the string theory determines the absolute values of $U(1)_X$ charges with $u^2 = \frac{1}{40}$ from $g_5 = g_X$ at the compactification scale. With this unit, the bare value of the Weinberg angle is also determined from Eq. (104):
\[
\sin^2 \theta_W^0 = \frac{3}{8}.
\] (108)

7.2. Embedding of $Y$ in flipped SU(5)

A covariant derivative in flipped SU(5) includes the term
\[
\sqrt{\frac{3}{5}} Y_5 g_5 A_5 + X u g_X A_x,
\] (109)
where $Y_5 = \text{diag}(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{2})$ and $X \equiv x \cdot \text{diag}(1, 1, 1, 1)$ denotes the $U(1)_X$ charge operator employed in this paper, $u$ a unit of $U(1)_X$ charge. So far $u$ is tacitly assumed to be unity, but generically it is not necessarily 1. $\sqrt{\frac{3}{5}} Y_5$ is one of SU(5) generator normalized with $\text{Tr} T_{SU(5)} T_{SU(5)} = \frac{1}{2}$. $A_5$ and $A_x$ stand for the SU(5) and U(1)$_X$ gauge fields, respectively. After symmetry breaking, one linear combination of $A_5$ and $A_x$ becomes the $U(1)_Y$ gauge field $B_\mu$ of the standard model. We introduce a mixing angle $\phi$,
\[
\begin{pmatrix}
A_5 \\
A_x
\end{pmatrix} = \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix} \begin{pmatrix}
B_\mu \\
C_\mu
\end{pmatrix},
\] (110)

Eq. (109) is recast in the following form,
\[
\left[\frac{1}{5} Y_5 + \frac{1}{5} X \sqrt{\frac{3}{5}} g_X/3 g_5 \tan \phi u\right] \sqrt{15} \cos \phi g_5 B_\mu + [C_\mu \text{ terms}],
\] (111)
where $C_\mu$ achieves a superheavy mass from the 10-dimensional Higgs VEVs $\langle \nu^c_H \rangle$, $\langle \bar{\nu}^c_H \rangle$. From this expression, one can read off the $U(1)_Y$ charges and coupling,

$$Y = \frac{1}{5} Y_5 + \frac{1}{5} X \sqrt{\frac{5}{3}} g_X \frac{g_Y}{g_5} \tan \phi u,$$

$$\frac{g_Y}{g_5} = \sqrt{\frac{15}{3}} \cos \phi.$$

Equation (112)

Since $Y(\nu^c) = 0$ is the SM direction, Eq. (112) should be

$$Y = \frac{1}{5} Y_5 + \frac{1}{5} X,$$

or

$$\frac{g_5}{g_X} = \sqrt{\frac{5}{3}} \tan \phi u.$$

With Eqs. (113) and (115), the relation between $g_5$, $g_X$, and $g_Y$ is derived:

$$g_Y^{-2} = \frac{g_5^{-2}}{15} + \frac{g_X^{-2}}{25u^2}.$$

8. Conclusion

We constructed a supersymmetric flipped $SU(5)$ model from a $Z_{12}$ orbifold compactification. The notable features of the model are:

- From $E_8$, the only non-Abelian group is the needed $SU(5)$. This is possible only for one shift vector in $Z_{12}$. In this sense, it is the unique $Z_{12}$ flipped $SU(5)$ model.
- Three families are obtained. The third family is located in twisted sectors while the first two families are in the untwisted sector. Separating the third family from the first two families enables a mass hierarchy of fermions.
- There exists a doublet–triplet splitting from a kind of the missing partner mechanism.
- There results only one pair of Higgs doublets.
- The $R$-parity is present.
- Allowed Yukawa couplings can generate a GUT scale VEVs of $10^H$ and $\bar{10}^H$ for the GUT breaking down to the standard model.
- There exist $Q_{em} = \pm \frac{1}{2}$ particles. But these form vector-like representations and most are removed at the GUT scale. Thus, $\sin^2 \theta_W$ is similar to that of $SO(10)$ GUT.

In this paper, all the relevant Yukawa couplings are derived from string construction. So far, we have not encountered any serious phenomenological problem. Successful Yukawa couplings may be generated by appropriate GUT scale VEVs of singlet fields which are treated here as free parameters. Finally, it is expected that a standard model can be derived from $Z_{12}$ without going through the intermediate stage of the flipped $SU(5)$ with features discovered in the present model [44]. In a future communication, we will tabulate all computer-searched $Z_{12}$ orbifold models.

Acknowledgements

We thank Ji-hun Kim for numerical checking of the spectra, and thank K.-S. Choi and I.-W. Kim for useful discussions. One of us (J.E.K.) also thank the Kavli Institute for Theo-
Appendix A. Flipped $SU(5)$ with $SU(4)'$

One can eliminate some exotic particles carrying $Q_X = \pm \frac{1}{2}, \pm \frac{5}{2}$ observed in the model discussed in the main text (Model I) by employing more complicated shift vector and Wilson line,

$$V = \left( \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{5}{12}, \frac{6}{12}, 0 \right) \left( \frac{2}{12}, \frac{2}{12}; 1, 0; 0^{\dagger} \right)' ,$$

$$a_3 = \left( 0, 0, 0, 0, 0; 0; -1 \frac{1}{3}, \frac{1}{3} \right) \left( 0, 0; 0; \frac{2}{3}, \frac{2}{3}; -1 \frac{1}{3}, -1 \frac{1}{3} \right)' ,$$

which satisfy the modular invariance conditions. Model II, constructed with these shift vector and Wilson line, eliminates in particular all $\mathbf{5}_{1/2}$ and $\mathbf{\bar{5}}_{-1/2}$ from the massless spectrum, leaving intact the MSSM fields obtained in Model I. Here, the gauge group is further broken down to

$$\left[ \{ SU(5) \times U(1)_X \} \times U(1)^3 \right] \times \left[ SU(4) \times SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times U(1)^2 \right]' .$$

Model II gives the same spectrum as Model I for the $T_6$, $T^0_2$, and $T^0_4$ sectors and visible sector of $U$ in Model I. As in Model I, there is no massless states satisfying $(P + 3V) \cdot a_3 = 0 \mod Z$ in the $T_3$ sector. The spectrum of Model II is summarized as follows:

- Fields of flipped $SU(5)$: $3 \times \mathbf{16}_{\text{flip}} + 1 \times \{ \mathbf{5}_{-2}, \bar{\mathbf{5}}_2, \} + 1 \times \{ \mathbf{10}_1, \mathbf{\bar{10}}_{-1} \}$, where $\mathbf{16}_{\text{flip}} \equiv \{ \mathbf{10}_1, \mathbf{\bar{3}}_3, \mathbf{1}_5 \}$.
- (Regularly charged) vector-like fields: $2 \times \{ \mathbf{16}_{\text{flip}}, \mathbf{\bar{16}}_{\text{flip}} \} + 2 \times \mathbf{10}_{\text{flip}}$, where $\mathbf{10}_{\text{flip}} \equiv \{ \mathbf{5}_{-2}, \bar{\mathbf{5}}_2 \}$.

### Table 11

<table>
<thead>
<tr>
<th>$P \cdot V$</th>
<th>$\tilde{s}$, $U_j$</th>
<th>Visible states</th>
<th>$SU(5) \times SU(1)_X$</th>
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<tbody>
<tr>
<td>$\frac{4}{T2}$</td>
<td>$(-+-+), U_3$</td>
<td>($+-++: +++$)</td>
<td>$\mathbf{5}_L^3$</td>
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<td></td>
<td></td>
<td>($+++-: --+$)</td>
<td>$\mathbf{10}_L^{-1}$</td>
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<td></td>
<td></td>
<td>($++++: +++$)</td>
<td>$\mathbf{1}_L^5$</td>
</tr>
<tr>
<td>$\frac{5}{T2}$</td>
<td>$(-+-), U_2$</td>
<td>($-1, 0, 0, 0; 0, 0, 0$)</td>
<td>$\mathbf{5}_L^2$</td>
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<tr>
<td></td>
<td></td>
<td>($-+++: ---$)</td>
<td>$\mathbf{10}_L^R$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($-++++: +++$)</td>
<td>$\mathbf{1}_L^5$</td>
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</table>

<table>
<thead>
<tr>
<th>$P \cdot V$</th>
<th>$\tilde{s}$, $U_j$</th>
<th>Hidden states</th>
<th>$[SU(4) \times SU(2)^2]'$</th>
</tr>
</thead>
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<tr>
<td>$\frac{4}{T2}$</td>
<td>$(-+-+), U_2$</td>
<td>($-; ++; \pm \pm \mp \mp$), ($-; -; \pm \pm \pm$)</td>
<td>$(6, 2, 1, 1)_L'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>($-; ++; \pm \pm \mp \mp$), ($-; -; \pm \pm \pm$)</td>
<td>$(6, 2, 1, 1)_L'$</td>
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<tr>
<td></td>
<td></td>
<td>($-; ++; \pm \pm \mp \mp$), ($-; -; \pm \pm \pm$)</td>
<td>$(6, 2, 1, 1)_L'$</td>
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<tr>
<td>$\frac{4}{T2}$</td>
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<td>($1.1; 0, 0, 0, 0, 0$)</td>
<td>singlet</td>
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</tbody>
</table>
Table 12
Massless states satisfying \( (P + 6V) \cdot a_3 = 0 \mod Z \) in \( T_6 \). The definitions of \( 5_3, \overline{5}_3, 10_1, \overline{10}_1, \) and \( 1_{ \pm 5,0} \) are found in the main text.

<table>
<thead>
<tr>
<th>( P + 6V )</th>
<th>( (N^L)_j )</th>
<th>( \Theta_0 )</th>
<th>( \mathcal{P}_6 )</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (53; \frac{1}{2}, 0, 0)(0^8)' )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (53; \frac{1}{2}, 0, 0)(0^8)' )</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>( L )</td>
</tr>
<tr>
<td>( (1-5; \frac{1}{2}, 0, 0)(0^8)' )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>3</td>
<td>( L )</td>
</tr>
<tr>
<td>( (5-3; \frac{1}{2}, 0, 0)(0^8)' )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>4</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, 0, 0)(0^8)' )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>5</td>
<td>( L )</td>
</tr>
<tr>
<td>( (15; \frac{1}{2}, 0, 0)(0^8)' )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>6</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2})(0^8)' )</td>
<td>13</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2})(0^8)' )</td>
<td>13</td>
<td>( \frac{1}{2} )</td>
<td>2</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2})(0^8)' )</td>
<td>13</td>
<td>( \frac{1}{2} )</td>
<td>3</td>
<td>( L )</td>
</tr>
</tbody>
</table>

Table 13
Chiral matter fields satisfying \( \Theta_0, +, - = 0 \) in the \( T_2^{0,+,−} \) sectors.

<table>
<thead>
<tr>
<th>( P + 2V )</th>
<th>( (N^L)_j )</th>
<th>( \mathcal{P}_2(f_0) )</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (53; \frac{1}{6}, 0, 0)(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (53; \frac{1}{6}, 0, 0)(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>2</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>2, 3</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>2, 3</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>2, 3</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>2, 3</td>
<td>1</td>
<td>( L )</td>
</tr>
</tbody>
</table>

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<th>( P + 2V )</th>
<th>( (N^L)_j )</th>
<th>( \mathcal{P}_2(f⁺) )</th>
<th>( \chi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
</tbody>
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<tr>
<th>( P + 2V )</th>
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<th>( \chi )</th>
</tr>
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<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
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<td>( L )</td>
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<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
<tr>
<td>( (10_1; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(\frac{1}{2}, \frac{1}{2}, 0^6)' )</td>
<td>0</td>
<td>1</td>
<td>( L )</td>
</tr>
</tbody>
</table>
• Exotic particles: $16 \times \{15/2, 1, -5/2\}$.
• A lot of neutral singlets under $SU(5) \times U(1)_Y$.

The full massless spectrum of Model II is presented in Tables 11–15.

In Table 13, the abbreviated symbols denote

$$(\bar{4}, 1, 1, 2)_{2^+} \equiv (\pm \frac{2}{3}, \pm \frac{1}{3}; 0, 0; -\frac{2}{3}, \frac{1}{3}, \frac{1}{3})',$$

$$(1, 1, 2, 2)_{2^+} \equiv (\pm \frac{1}{6}, \pm \frac{1}{6}; \pm \frac{1}{3}, 0; -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6})',$$

$$(1, 2, 1, 2)_{2^+} \equiv (\pm \frac{1}{6}, \pm \frac{1}{6}; \pm \frac{1}{3}, 0; -\frac{1}{6}, \frac{1}{6}, -\frac{1}{6})',$$

$$(1, 1, 1, 1)_{2^+} \equiv (\frac{1}{3}, \frac{1}{3}; 0, 0; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})'.$$

Table 14:
Chiral matter fields in the $T^0_4, T^+_4$ and $T^-_4$ sectors. In the $T^+_4$ and $T^-_4$ sectors, $(1, 2, 1, 1)'_{4^+} \equiv (\frac{1}{6}, \frac{1}{6}; \pm \frac{1}{6}, \frac{1}{6})$, $(1, 1, 1, 1)'_{4^+} \equiv (\frac{1}{6}, \frac{1}{6}; \pm \frac{1}{6}, \frac{1}{6})$, $(1, 2, 1, 1)'_{4^-} \equiv (\frac{1}{6}, \frac{1}{6}; \pm \frac{1}{6}, \frac{1}{6})'$ and $(1, 1, 1, 1)'_{4^-} = (\frac{1}{6}, \frac{1}{6}; 0, 0; \frac{1}{3}, \frac{1}{3}, \frac{1}{3})'$.
Table 15
Chiral matter fields satisfying θ_{0,0} = 0 in the T_1^0, T_1^+, T_S^0, and T_S^+ sectors. There are no massless states in T_1^- and T_S^-.

Here, 1_{5/2} \equiv (±\frac{1}{3}, ±\frac{1}{3}, ±\frac{1}{3}, ±\frac{1}{3}, ±\frac{1}{3}, ±\frac{1}{3} )',

(1, 1, 2, 1)_{1+}^+ = (±\frac{1}{6}, ±\frac{1}{3}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6} )',

(1, 1, 2, 1)_{1+}^+ = (±\frac{1}{6}, ±\frac{1}{3}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6} )',

(1, 1, 2, 1)_{1+}^+ = (±\frac{1}{6}, ±\frac{1}{3}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6} )'.

(\bar{\mathbf{4}}, 1, 2, 1)_{2+}^+ = (±\frac{1}{6}, ±\frac{1}{3}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6} )',

(1, 1, 1, 1)_{2+}^+ = (±\frac{1}{6}, ±\frac{1}{3}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6} )',

(1, 2, 1, 1)_{2+}^+ = (±\frac{1}{6}, ±\frac{1}{3}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6} )',

(1, 1, 1, 1)_{2+}^+ = (±\frac{1}{6}, ±\frac{1}{3}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6}, ±\frac{1}{6} )'.

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2006, Appendix D.

1985, p. 276;


