Efficient quantum computation using coherent states

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We study universal quantum computation using optical coherent states. A teleportation scheme for a coherent-state qubit is developed and applied to gate operations. This scheme is shown to be robust to detection inefficiency.

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I. INTRODUCTION

The theory of quantum computation promises to revolutionize the future of computer technology in factoring large integers [1] and combinational searches [2]. In recent years, the physical implementation of a quantum computer has been intensively studied. Quantum computing in optical systems has been studied as one of several plausible models. Recently, Knill et al. suggested a scheme for efficient quantum computation with linear optics [3].

A coherent field is a fundamental tool in quantum optics and linear superposition of two coherent states is considered one of the realizable mesoscopic quantum systems [4]. In particular, Cochrane et al. [5] showed how logical qubits can be implemented using even and odd coherent superposition states which are defined as \( N^0_{\pi}(|\alpha|\pm|\alpha|) \) with \(|\alpha|\) and \(|-\alpha|\) representing coherent states of \( \pi \) phase difference space and \( N^0_{D} \) being normalization factors. The two superposition states form orthogonal bases in two-dimensional Hilbert space and they can be discriminated by photon measurement [6]. There were some proposals to entangle the logical qubits with atomic states [7]. One drawback of using even and odd cat states as a logical qubit basis for quantum computation is that they are extremely sensitive to photon loss and detection inefficiency.

In this paper we present a method to implement universal quantum computation using coherent states. This proposal makes it possible to realize universal quantum computation based on quantum teleportation [8] which was shown to be a useful tool in the controlled gate operation [9]. It is also found that this scheme is robust to detection inefficiency.

II. READOUT SCHEME AND UNIVERSAL GATE OPERATIONS

Let us consider two coherent states \(|\alpha\rangle\) and \(|-\alpha\rangle\), where the coherent amplitude \(\alpha\) is taken to be real. The two coherent states are not orthogonal to each other but their overlap \(\langle \alpha|-\alpha\rangle=e^{-2\alpha^2}\) decreases exponentially with \(\alpha\). For example, when \(\alpha\) is as small as 3, the overlap is \(\approx 10^{-8}\). Throughout the paper, the average photon number of the coherent state is assumed around 10. We identify the two coherent states of \(\alpha\) as basis states for a logical qubit,

\[
|\alpha\rangle \rightarrow |0_L\rangle, \quad |\alpha\rangle \rightarrow |1_L\rangle.
\]  

A qubit state is then represented by \(|\phi\rangle = A|\alpha\rangle + B|-\alpha\rangle\), where the normalization condition is

\[
1 = \langle \phi|\phi\rangle = |A|^2 + |B|^2 + \langle AB^* + A^*B\rangle|\alpha|-\alpha\rangle
\]

\[
\]  

Let us consider the readout of a qubit. The logical basis states, \(|\alpha\rangle\) and \(|-\alpha\rangle\), can be discriminated by a simple measurement scheme with a 50-50 beam splitter, an auxiliary coherent field of amplitude \(\alpha\), and two photodetectors as shown in Fig. 1. At the beam splitter, the input state \(|\phi\rangle_1\) is superposed with the auxiliary state \(|\alpha\rangle_2\) and gives the output

\[
|\phi_R\rangle_{ab} = A\sqrt{2}\alpha|\alpha\rangle_a|0\rangle_b + B|\alpha\rangle_a|\alpha\rangle_b - \sqrt{2}\alpha|\alpha\rangle_b.
\]  

If detector \(A\) registers any photon(s) while detector \(B\) does not, we know that \(|\alpha\rangle\) is measured. On the contrary, if \(A\) does not click while \(B\) does, the measurement outcome is \(|-\alpha\rangle\). Even though there is a nonzero probability of failure \(P_f = 0.5(0|\alpha\rangle_2\langle 0|\phi_R\rangle_{ab}|0\rangle_a|\alpha\rangle_b - 1\) in which both of the detectors do not register a photon, the failure is known from the result whenever it occurs, and \(P_f\) approaches zero exponentially as \(\alpha\) increases.

An arbitrary 1-bit rotation and a controlled-NOT (c-NOT) gate for two-qubit states form a set which satisfies all the requirements for a universal gate operation. For any SU(2) unitary operation, there is a unique rotation \(R(\theta, \phi, \eta)\) around the \(x\), \(y\), and \(z\) axes. Cochrane et al. showed that the rotation around the \(x\) axis for even and odd coherent superposition states can be realized using an interaction Hamiltonian \(H_D = \hbar(\beta a^2 + B^*a)\), where \(\beta\) is the complex amplitude of the classical driving force [5]. The evolution by this Hamiltonian corresponds to the displacement operator.

\[
|\phi> = \begin{cases} 0, & \text{if detector } A \text{ registers any photon(s)} \\ |\alpha\rangle, & \text{if detector } B \text{ does not} \\ |\alpha\rangle, & \text{if detector } A \text{ does not and detector } B \text{ does} \end{cases}
\]  

FIG. 1. Measurement scheme for \(|\phi\rangle_1 = A|\alpha\rangle_1 + B|-\alpha\rangle_1\) with a 50-50 beam splitter and auxiliary state \(|\alpha\rangle_2\). If detector \(A\) registers any photon(s) while detector \(B\) does not, the measurement outcome is \(|\alpha\rangle\), i.e., \(|0_L\rangle\). On the contrary, \(A\) does not click while \(B\) does, the measurement outcome is \(|-\alpha\rangle\), i.e., \(|1_L\rangle\).
We can easily check their similarity by calculating the fidelity product $D$ (8). In a similar way, the displacement operator $D(\delta)$ suffices to make one cycle of rotation as $\delta=\pi/4$. It is known that the effect of the beam splitter is described by $D(i\mathcal{E}\sqrt{1-T})$ in the limit of $T\to 1$ and $\mathcal{E}\gg 1$. (More rigorously the output state becomes mixed but in the limit it can well be approximated to a pure state as shown by one of the authors [10].)

Thus the rotation angle $\theta$ depends on $\alpha$ and $\epsilon$: $\theta=4\alpha\epsilon$. A small amount of $\epsilon$ suffices to make one cycle of rotation as $\alpha$ is relatively large. The displacement operation $D(i\epsilon)$ can be effectively performed using a beam splitter with the transmission coefficient $T$ close to unity and a high-intensity coherent field of amplitude $i\mathcal{E}$, more shown in Fig. 2(a). It is known that the effect of the beam splitter is described by $D(i\mathcal{E}\sqrt{1-T})$ in the limit of $T\to 1$ and $\mathcal{E}\gg 1$. (More rigorously the output state becomes mixed but in the limit it can well be approximated to a pure state as shown by one of the authors [10].)

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state [13,14] including an entanglement purification scheme [14]. However, the success probability of this teleportation scheme is limited to less than 1/2 in practice and the required photon parity measurement is very sensitive to detection inefficiency and photon loss as the parity alternates by missing one photon. We suggest a teleportation protocol as follows to circumvent those problems.

For any ideal teleportation scheme, a maximally entangled pair, Bell measurement and unitary operations are required [8]. In our case, necessary unitary operations \( \sigma_x \) and \( \sigma_z \) correspond to a phase shift \( P(\pi) \) and displacement operation \( D(i\pi/4\alpha\sqrt{1-T}) \), respectively. An entangled coherent channel \( |\Phi_+\rangle = N_+ ((|\alpha\rangle |\alpha\rangle + |\alpha\rangle |\alpha\rangle)) \), where \( N_+ \) is a normalization factor, can be generated from a coherent state passing through an \( H_\pi \) gate and a 50-50 beam splitter as shown in Fig. 4(a). The superscript \( \sqrt{2} \) in \( H_\pi \) stands for the amplitude of the incident field being \( \sqrt{2}\alpha \). Note that the coherent amplitude \( \Delta \alpha \) for a unitary operation shown in Fig. 3 should be \( i\pi/[8\alpha\sqrt{2(1-T)}] \) for the \( H_\pi \) gate operation. The Bell measurement shown in Fig. 4(b) is to distinguish four quasi-Bell states [15],

\[
|\Phi_\pm\rangle = N_\pm (|\alpha\rangle |\pm \alpha\rangle + |\pm \alpha\rangle |\alpha\rangle),
\]

\[
|\Psi_\pm\rangle = N_\pm (|\alpha\rangle |\alpha\rangle + |\alpha\rangle |\alpha\rangle),
\]

where \( |\pm \alpha\rangle = |\pm \alpha\rangle \otimes |\pm \alpha\rangle \). Note that the quasi-Bell states become maximally entangled Bell states when \( \alpha \) is large. If the incident field to the first beam splitter in Fig. 4(b) is \( |\Phi_+\rangle_{12} \), it becomes \( |0,2\alpha, -\sqrt{2\alpha}, \sqrt{2\alpha}\rangle_{abcd} \) at detectors \( A, B, C, \) and \( D \). If detector \( A \) does not click while the others do, the measurement outcome is \( |\Phi_+\rangle_{12} \). Likewise, only \( B \) does not click for the measurement outcome \( |\Phi_+\rangle_{12} \), \( C \) for \( |\Psi_+\rangle_{12} \), and \( D \) for \( |\Psi_-\rangle_{12} \). The failure probability for which no photon is detected at more than one detector, which is due to the nonzero probability of \( |0\pm 2\alpha\rangle \) and \( |0\pm \sqrt{2\alpha}\rangle \), approaches zero rapidly as \( \alpha \) increases, and, moreover, the failure is always known when it occurs. The scheme to teleport \( |\phi\rangle \) via the entangled channel \( |\Phi_+\rangle \) is summarized in Fig. 4(c). When the Bell measurement outcome is \( |\Phi_+\rangle \), the output state does not need any operation. When the Bell measurement outcome is \( |\Phi_-\rangle \) or \( |\Psi_\pm\rangle \), \( \sigma_z \) or \( \sigma_x \) is required, respectively. The unitary operations \( \sigma_z \) and \( \sigma_x \) should be successively applied for the output \( |\Psi_\pm\rangle \).

Gottesman and Chuang showed that the teleportation protocol can be used to construct a c-NOT gate [9]. To apply their suggestion in our scheme, we need to use two three-mode entangled states represented by

\[
|\xi\rangle = N(|\sqrt{2}\alpha, \alpha, \alpha\rangle + |\sqrt{2}\alpha, -\alpha, -\alpha\rangle),
\]

where \( N \) is a normalization factor, and the quantum teleportation protocol we just developed. The entangled state \( |\xi\rangle \) can be generated by passing a coherent field \( |2\alpha\rangle \) through a \( H^2 \) gate, which is a Hadamard gate for a qubit with logical bases \( |\pm 2\alpha\rangle \), and two 50-50 beam splitters as shown in Fig. 5(a). After generating \( |\xi\rangle_{abc} \) and \( |\xi\rangle_{def} \), Hadamard operations are applied to \( |\xi\rangle_{def} \) as shown in Fig. 5(a). This makes the given state \( |\xi\rangle_{abc} \otimes |\xi\rangle_{def} \) to be
where approximation (2) is used. For example, suppose that 
\( \alpha = 3 \) and the detection efficiency of the detectors is 90% which is a reasonable value for an avalanched photodetector [16], the failure probability \( P_f \) that the detector misses all the photons is 
\( P_f \approx 9 \times 10^{-8} \).

If the effect of \( \epsilon \) for the displacement operator is not negligible, a qubit state \( |\phi^+\rangle = D(i\epsilon_1)\cdots D(i\epsilon_N)|\phi\rangle \) after \( N \) displacement operations may be

\[
|\phi^+\rangle = \mathcal{A}'\left|\alpha + i\sum_{n=1}^{N} \epsilon_n\right\rangle + B'\left|-\alpha + i\sum_{n=1}^{N} \epsilon_n\right\rangle .
\]

(18)

After passing a 50-50 beam splitter with an auxiliary state \( |\alpha\rangle_2 \) as shown in Fig. 1, the state \( |\phi^+\rangle \) becomes

\[
|\phi^+_R\rangle_{ab} = \mathcal{A}'\left|\sqrt{2}\alpha + \frac{i\epsilon}{\sqrt{2}}\right\rangle_1\left|\frac{i\epsilon}{\sqrt{2}}\right\rangle_2 + B'\left|-\sqrt{2}\alpha + \frac{i\epsilon}{\sqrt{2}}\right\rangle_1\left|\frac{i\epsilon}{\sqrt{2}}\right\rangle_2 .
\]

(19)

In this condition, there is nonzero probability \( P_f^d \) for undetected errors in which detector \( A(B) \) detects any photon and \( B(A) \) does not while the incident state \( |\phi^+\rangle \) was \( |1\rangle_1 (|0\rangle_2) \) (see Fig. 1). For the worst case, all \( \epsilon_n \)'s may have the same sign with a large \( N \). One useful trick to overcome this problem is to flip the sign of \( \epsilon_n \) appropriately for each operation, noting that the rotation \( R(\theta) \) can be performed both by positive and negative \( \theta \). In this way, we can keep \( \sum_{n=1}^{N} \epsilon_n \sim \bar{\epsilon} = \pi/4 \alpha \), regardless of \( N \), then Eq. (19) can be represented as

\[
|\phi^+_R\rangle_{ab} = \mathcal{A}'\left|\sqrt{2}\alpha + \frac{i\epsilon}{\sqrt{2}}\right\rangle_1\left|\frac{i\epsilon}{\sqrt{2}}\right\rangle_2 + B'\left|-\sqrt{2}\alpha + \frac{i\epsilon}{\sqrt{2}}\right\rangle_1\left|\frac{i\epsilon}{\sqrt{2}}\right\rangle_2 .
\]

(20)

In this condition, the fidelity between the final state (18) and the ideal output is proportional to \( e^{-2\bar{\epsilon}} \) from Eq. (5). Fidelity of \( \sim 0.93 \) is then obtained for \( \alpha = 3 \).

Differently from \( P_f^d \), the undetected error probability \( P_f^d \) is a probability of making an error without being recognized. Considering the accumulated error as in Eq. (20), in order to minimize the undetected error \( P_f^d \) while keeping \( P_f^d \) low, we need to modify the criterion to discriminate \(|\pm \sqrt{2}\alpha + i\bar{\epsilon}\sqrt{2}\rangle\) and \(|\pm \bar{\epsilon}\sqrt{2}\rangle\). Ideally we took \( \bar{\epsilon} = 0 \) and discriminated the two states by detection of any photons and no photon. In this case, the probability of \(|\pm \sqrt{2}\alpha + i\bar{\epsilon}\sqrt{2}\rangle\) registering no photon is

\[
p_A = \sum_{n=0}^{\infty} |n| |\sqrt{2}\alpha + i\bar{\epsilon}\sqrt{2}|^2 (1-d)^n (1-d)^m
\]

(21)

and the probability of the state \(|i\bar{\epsilon}/\sqrt{2}\rangle\) registering one or more photons is

\[
p_B = \sum_{n=m}^{\infty} \sum_{m=1}^{\infty} |n| |i\bar{\epsilon}/\sqrt{2}|^2 |C_m| d^n (1-d)^n (1-d)^m
\]

(22)

where \( nC_m = n!/m!(n-m)! \). Both \( p_A \) and \( p_B \) approach to zero as \( \alpha \) increases. We then obtain undetected error prob-
ability $P_d^f = p_A \times p_B$. On the other hand, the success probability $P_s$ is the probability in that $|i \hat{e}/\sqrt{2}\rangle$ yields no photon and $\pm |\sqrt{2}\alpha + i \hat{e}/\sqrt{2}\rangle$ yields any photon(s):

$$P_s = \sum_{n=0}^{\infty} |\langle n | i \hat{e}/\sqrt{2}\rangle|^2 (1 - d)^n + \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} |\langle n | \sqrt{2}\alpha + i \hat{e}/\sqrt{2}\rangle|^2 (1 - d)^{n-m}. \quad (23)$$

The detected error probability is $P_d^e = 1 - P_s - P_f^e$. Suppose that $\alpha = 3$ ($\hat{e}$ is then $\approx 0.26$), and the detection efficiency is again 90%, $p_A \approx 9 \times 10^{-8}$ and $p_B \approx 0.03$ are obtained. If we keep the criterion for the ideal case, we find $P_d^e \approx 3 \times 10^{-9}$ and $P_d^f \approx 0.030$. However, if we take the registration of 0, 1, and 2 photons as the measurement of $|i \hat{e}/\sqrt{2}\rangle$ then $p_A$, $p_B$, and $P_s$ should be redefined as follows:

$$p_A = \sum_{n=0}^{\infty} |\langle n | i \hat{e}/\sqrt{2}\rangle|^2 (1 - d)^n + \sum_{n=1}^{\infty} |\langle n | \sqrt{2}\alpha + i \hat{e}/\sqrt{2}\rangle|^2 (1 - d)^{n-1}$$

$$+ \sum_{n=2}^{\infty} |\langle n | \sqrt{2}\alpha + i \hat{e}/\sqrt{2}\rangle|^2 (1 - d)^{n-2}, \quad (24)$$

$$P_s = \sum_{n=0}^{\infty} \sum_{m=n}^{\infty} |\langle n | i \hat{e}/\sqrt{2}\rangle|^2 d^n (1 - d)^n + \sum_{n=1}^{\infty} |\langle n | \sqrt{2}\alpha + i \hat{e}/\sqrt{2}\rangle|^2 d^n (1 - d)^{n-1}$$

$$+ \sum_{n=2}^{\infty} |\langle n | \sqrt{2}\alpha + i \hat{e}/\sqrt{2}\rangle|^2 d^n (1 - d)^{n-2} \times (1 - d)^{n-m}. \quad (26)$$

We then find $P_d^e \approx 0.030$. However, if we take the registration of 0, 1, and 2 photons as the measurement of $|i \hat{e}/\sqrt{2}\rangle$ then $p_A$, $p_B$, and $P_s$ should be redefined as follows:

$$P_s = \sum_{n=0}^{\infty} |\langle n | i \hat{e}/\sqrt{2}\rangle|^2 (1 - d)^n + \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \sum_{n=m}^{\infty} |\langle n | \sqrt{2}\alpha + i \hat{e}/\sqrt{2}\rangle|^2 d^n (1 - d)^{n-m}. \quad (26)$$

The energy decay rate $\gamma$ is the interaction time, and $\mathcal{N}_g$ is a normalization factor. Considering decoherence, we need to change $|0\rangle$ and $|1\rangle$ to $|\alpha\rangle$ and $|\bar{\alpha}\rangle$. The auxiliary coherent fields for computation have to be changed likewise. The larger the initial coherent amplitude $\alpha$ is, the longer the condition that $\langle \alpha | - \bar{\alpha} \rangle = 0$ is preserved, but the decoherence becomes more rapid as $\alpha$ increases because $\Gamma$ decreases more rapidly for a larger $\alpha$. The energy decay rate $\gamma$ of the relevant system and number of required operations for computation may be the crucial factors to decide the value of $\alpha$. However, decohered states can still be represented by combinations of 1-bit errors for time-dependent logical qubits $|\alpha\rangle$ and $|\bar{\alpha}\rangle$. It is known that an error correction circuit for an arbitrary 1-qubit error can be built using c-NOT and 1-bit unitary operations [18].

IV. REMARKS

In conclusion, we have found that near-deterministic universal quantum computation can be realized using coherent states. Efficient readout is possible using beam splitters and coherent light sources. Single-bit unitary transformation can be performed using beam splitters and nonlinear media, and a c-NOT gate can be constructed based on teleportation protocol. Teleportation of a coherent state qubit can be accomplished with a complete Bell measurement for a large coherent amplitude using nonlinear media, photodetectors, coherent light sources, and beam splitters. Decohered states can be represented by combinations of 1-bit errors for time-dependent coherent state qubits of reduced amplitude. A purification scheme for decohered entangled channels has been studied [19]. Detailed error correction methods for our scheme deserve further investigation. The nonlinear effect [4] used in this paper is typically too weak to generate the required superposition states in current technology. The study of generating a coherent superposition of optical states requires further study.

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[6] V. Bužek and P.L. Knight, in Progress in Optics XXXIV, ed-


