Quantification of Macroscopic Quantum Superpositions within Phase Space

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Based on phase-space structures of quantum states, we propose a novel measure to quantify macroscopic quantum superpositions. Our measure simultaneously quantifies two different kinds of essential information for a given quantum state in a harmonious manner: the degree of quantum coherence and the effective size of the physical system that involves the superposition. It enjoys remarkably good analytical and algebraic properties. It turns out to be the most general and inclusive measure ever proposed that it can be applied to any types of multipartite states and mixed states represented in phase space.

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Quantum superposition is often considered the most crucial feature of quantum mechanics. Its evidence has been witnessed in numerous experiments using microscopic physical systems. However, the question of whether a truly macroscopic system could ever be in a quantum superposition involves far more nontrivial issues in both practical and philosophical aspects [1]. A macroscopic quantum superposition is supposed to consist of two (or more) macroscopically distinct states but still maintains certain potential to manifest quantum interference between the distinct component states.

Regarding the implementation of macroscopic quantum superpositions, limited but interesting progress has been made in atomic or molecular systems [2,3], superconducting circuits [4,5], and optical setups [6–8]. In particular, superpositions of coherent states (SCSs) [6], multimode Greenberger-Horne-Zeilinger (GHZ) states [7], and NOON states [8] have been experimentally demonstrated in optical systems. Interestingly, certain types of ''bigger-size'' superpositions may be useful for quantum information processing [9]. However, even though various types of macroscopic superpositions have been theoretically studied and some experimental success made, the definition of their measure applicable to all those states has remained a difficult yet urgent task.

Attempts to find a good measure for macroscopic quantum superpositions date back to Leggett [10]. It has been followed by several proposals [11–15] and each of them has its own merit and insight. In those proposals, people often start from considering the effective number of particles that involve the superposition [10,11,15]. It could also be natural to take notice of the distance between the component states [11–15]. Typically, these measures also depend on the choice of a specific target state [11,14], or a decomposition or observable [12,13,15].

First of all, it is crucial to note that the number of effective particles (or distance between the component states) cannot witness the true quantum superposition when determining the size of a macroscopic superposition. These factors do not allow one to conclusively discriminate between a coherent superposition and a classical mixture, not to mention partially mixed states. This problem was also pointed out in Ref. [13] where a signature of a macroscopic superposition was studied. This is unignorable because macroscopic superpositions typically lose quantum coherence, at least to some extent, due to interactions with their environments; this process is called decoherence [16]. In other words, a proper measure for a macroscopic superposition must quantify the degree of a true superposition against an incoherent mixture, together with its effective size factor such as the effective number of particles.

Furthermore, it should be pointed out that the choice of a target state or of a fiducial decomposition or observable, which the aforementioned measures employ, is actually arbitrary. For example, a SCS should not be accused of being less like a macroscopic superposition due to the reason that it does not look like a GHZ state, and vice versa. This problem, together with the first one mentioned above, causes the previous measures [10–15] to be limited to specific types of superpositions, and/or obscures comparisons between various types of states. For example, it would be difficult to compare a GHZ state and a continuous-variable Gaussian state, and even worse when both the states are somehow partially decohered. In order to effectively compare different types of states in terms of their sizes as macroscopic superpositions, a decomposition-independent (and measurement-independent) measure that can be commonly applied to any given state is highly desired.

In this Letter, we propose a novel measure that satisfies these requirements based on quantum interference in phase space. For an arbitrary given state in phase space, it provides quantitative information about both the crucial aspects of macroscopic superpositions: the effective size of the physical system that involves the superposition and the degree of quantum coherence. The appropriateness, inclusiveness, and usefulness of our proposal are confirmed by (i) its direct relation to a well-known decoherence model, (ii) its direct relation to a previous measure [11] proposed for a specific type of states, (iii) its advantageousness in computability as a practical tool, and (iv) various examples including mixed states with sensible results.

Among the previously proposed measures, Björk *et al.*'s one [12] is based on interference between component states, while it does not distinguish a pure superposition from a classical mixture. We attempt to consider quantum interference of a given state in a more general framework using a phase-space formalism. Phase-space representations such as the Wigner function are very useful to visualize a quantum state, from which some crucial information can be intuitively obtained. In terms of the Wigner function, a macroscopic quantum superposition has two (or more) well-separated peaks and has some oscillating patterns between them in phase space. It is known that these interference fringes tend to appear more frequent as the distinguishable peaks are more separate. We pay attention to the "frequency" of the interference as an indicator of a macroscopic superposition. Of course, it is a separate problem to quantify it in the phase-space structure for a proper measure.

The characteristic function for a density operator ρ for a single-mode case is defined as $\chi(\xi) = \mathrm{Tr}\{\rho \exp[\xi \hat{a}^\dagger - \xi^* \hat{a}]\}$ where $\hat{a}$ and $\hat{a}^\dagger$ are the bosonic annihilation and creation operators, respectively. The Wigner function $W(\alpha)$ is the Fourier transform of the characteristic function [17] as $W(\alpha_r, \alpha_i) = \frac{1}{\pi^2} \times \iint d\xi_r d\xi_i \chi(\xi_r, \xi_i) e^{-2i(\alpha_r \xi_i - \alpha_i \xi_r)}$, where subscript $r(i)$ denotes the real (imaginary) part of the given variable. We notice that a frequency of a Wigner-function component along the real (imaginary) axis is ξ_i (ξ_r) and its complex amplitude for specific frequency ξ corresponds to $\chi(\xi)$.

We know that (i) the frequency of the fringes (how dense the fringes are) reflects the "effective size" of the superposition (i.e., how far the component states separate), and (ii) "coherence" (i.e., the degree of genuine superposition against its completely mixed version, say, in terms of the "pointer basis" [16]) relates to the magnitude of the interference fringes. In order to quantify both the features at the same time, it is natural to take the sum over (size of frequency) $\times$ (absolute amplitude for the given frequency). Here, we take it in the form as $\int d^2\xi (\xi_r^2 + \xi_i^2) |\chi(\xi)|^2$, so that it quantifies both the frequency and the "magnitude" of interference fringes in the Wigner representation.

We present the formal definition of our interference-based measure as

$$\mathcal{I}(\rho) = \frac{1}{2\pi^M} \int d^2\boldsymbol{\xi} \sum_{m=1}^{M} [|\xi_m|^2 - 1] |\chi(\boldsymbol{\xi})|^2 \tag{1}$$

$$= \frac{\pi^M}{2} \int d^2\boldsymbol{\alpha} W(\boldsymbol{\alpha}) \sum_{m=1}^{M} \left[-\frac{\partial^2}{\partial \alpha_m \partial \alpha_m^*} - 1 \right] W(\boldsymbol{\alpha}), \tag{2}$$

where m indicates different modes, M the number of such modes, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_M)$, $\int d^2\boldsymbol{\xi} = \int d^2\xi_1 \int d^2\xi_2 \cdots \int d^2\xi_M$, and $\boldsymbol{\alpha}$ and $\int d^2\boldsymbol{\alpha}$ are defined in the same manner. The form of the above definition is based on several reasons that become clear in the remaining discussions. Since the above definition is grounded on a general characteristic of the Wigner function, it can be applied to any bosonic or multimode state not only for superpositions consisting of more than two components but also for any partially or fully mixed state.

A remarkable feature of $\mathcal{I}(\rho)$ is that it is directly related to a decoherence model as

$$\mathcal{I}(\rho) = -\mathrm{Tr}[\rho \mathcal{L}(\rho)], \tag{3}$$

where $\mathcal{L}(\rho)$ is the superoperator in the Lindblad form of a vacuum-environment decoherence model [17,18]:

$$\frac{d\rho}{d\tau} = \mathcal{L}(\rho) = \sum_{m=1}^{M} \left[\hat{a}_m \rho \hat{a}_m^\dagger - \frac{1}{2} \rho \hat{a}_m^\dagger \hat{a}_m - \frac{1}{2} \hat{a}_m^\dagger \hat{a}_m \rho \right], \tag{4}$$

where $\tau =$ (decayrate) $\times$ (time) is the dimensionless time. If we let $\mathcal{P} = \mathrm{Tr}(\rho^2)$ be the purity of state ρ, we find

$$\frac{d\mathcal{P}(\rho)}{d\tau} = -2\mathcal{I}(\rho). \tag{5}$$

Consequently, $\mathcal{I}(\rho)$ can be interpreted as the decreasing rate of the purity of ρ. This interpretation conforms exactly with one of Dür *et al.*'s [11] where the authors suggested a size measure for a specific type of superposition. The form of superposition studied in Ref. [11] is

$$|\phi\rangle = \mathcal{K}(|\phi_1\rangle^{\otimes N} + |\phi_2\rangle^{\otimes N}) \tag{6}$$

with $\mathcal{K}$ being the normalization factor and $|\langle\phi_1|\phi_2\rangle|^2 = 1 - \epsilon^2 \neq 0$ with small real value ϵ. Following Ref. [11], we take $|\phi_1\rangle = |0\rangle$ and $|\phi_2\rangle = \cos\epsilon|0\rangle + \sin\epsilon|1\rangle$ with assumptions $\epsilon^2 \ll 1$ and $N\epsilon^2 \gg 1$, and we obtain $\mathcal{I}(\rho) \simeq N\epsilon^2/4$ for state (6). Remarkably, this result is the same as the one in Ref. [11] only by a constant factor, even though our measure is derived from a starting point quite different from Ref. [11] where the effective particle number involving the superposition was concerned.

Along these lines, it is conjectured that even though our measure is based on the size of the frequency of interference fringes, it is closely related to the number of particles composing the superposition. For example, it can simply be shown from Eq. (3) that $\mathcal{I}(\rho)$ exactly gives the particle number n for a bosonic number state $|n\rangle$. Besides, $\mathcal{I}(\rho)$ properly assesses the degree of a true superposition against incoherent mixtures. The following theorem shows that only a pure state can give the maximum value of $\mathcal{I}(\rho)$ for a given average particle number. *Theorem:* $\mathcal{I}(\rho)$ has the maximum value $\langle\hat{n}\rangle$, the average number of particles for ρ, if and only if ρ is a pure state and is orthogonal to any one-particle-subtracted state of itself [19]. It follows that a mixed state always has a lower value of $\mathcal{I}(\rho)$ than its $\langle\hat{n}\rangle$.

It is straightforward to show that the maximum value of $\mathcal{I}(\rho) = \langle \hat{n} \rangle$ is obtained for the SCS $\propto |\alpha\rangle + |-\alpha\rangle$, where $|\pm\alpha\rangle$ are coherent states of amplitudes $\pm\alpha$, the GHZ state $\propto |0\rangle^{\otimes N} + |1\rangle^{\otimes N}$, and the NOON state $\propto |n\rangle|0\rangle + |0\rangle|n\rangle$. Note that the average particle number of the SCS is related to α as $\langle \hat{n} \rangle = \alpha^2 \tanh\alpha^2$, where α was assumed to be real without loss of generality. On the other hand, as shown above, mixed versions of the aforementioned states have values of $\mathcal{I}(\rho) < \langle \hat{n} \rangle$. We note that $\mathcal{I}(\rho) = 0$ for fully mixed states such as $\frac{1}{d} I_{d\times d}$, where $I_{d\times d}$ is a d-dimensional identity matrix, and $\rho \propto |\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|$. This means that no matter how large the size of the system is, if the state scarcely has potential for quantum interference, the measure $\mathcal{I}(\rho)$ gives the value close to zero.

Let us consider a partially mixed SCS under the decoherence effect caused by Eq. (4):

$$\rho = \mathcal{N}\{|t\alpha\rangle\langle t\alpha| + |-t\alpha\rangle\langle -t\alpha| + \Gamma(|t\alpha\rangle\langle -t\alpha| + |-t\alpha\rangle\langle t\alpha|)\}, \tag{7}$$

where $t = e^{-\tau/2}$, $\Gamma = \exp[-2(1-e^{-\tau})\alpha^2]$, and $\mathcal{N}$ is the normalization factor. Using Eq. (1), the interference-based measure is obtained as $\mathcal{I}(\alpha, \tau) = \langle \hat{n}(0) \rangle e^{-\tau} \sinh[2(2e^{-\tau} - 1)\alpha^2]/\sinh[2\alpha^2]$ where $\langle \hat{n}(0) \rangle$ denotes the average number of particles at $\tau = 0$. Here, the two crucial factors, the effective number of particles and the degree of true coherence, are properly measured by α and τ. In Fig. 1(a), we plot $\mathcal{I}(\rho)$ for several cases of SCSs against the normalized time $r = \sqrt{1 - e^{-\tau}}$. While SCSs with large amplitudes have large values of $\mathcal{I}(\rho)$, they decrease more rapidly than SCSs with small amplitudes. This is satisfactorily in accordance with the well-known fact, i.e., the rapid destruction of macroscopic quantum superpositions [16,20]. Here, a remarkable advantage of our measure is obvious that any fully or partially decohered superpositions are effectively quantified.

Our measure also provides sensible results for single-mode and multimode Gaussian continuous-variable states which are useful for quantum information applications [21]. As an example, a general form of the characteristic function for a single-mode Gaussian state is $\chi(\xi) = \exp[-A\xi_r^2/2 - B\xi_i^2/2]$. Real positive parameters A and B satisfy $AB \geq 1$ and the state is pure when the equality sign holds. Using Eq. (1), the interference-based measure is obtained as $\mathcal{I}(\rho) = (A + B - 2AB)/[4(AB)^{3/2}]$, and it is reduced to $(A + A^{-1} - 2)/4$ for pure states. Obviously, the more "squeezed" pure state ($A \gg 1$ or $A \approx 0$) gives the larger value of $\mathcal{I}(\rho)$, and it approaches infinity in the limit of the original Einstein-Podosky-Rosen state ($A \to \infty$ or $A \to 0$). Suppose that a pure Gaussian state ($A = e^{-2s}$ and $B = e^{2s}$), where s is the squeezing parameter, is under decoherence described by Eq. (4). The time-dependent state is then characterized by $A = r^2 + e^{-2s}t^2$ and $B = r^2 + e^{2s}t^2$. The measure $\mathcal{I}(\rho)$ is immediately obtained from the result above, which has been plotted in Fig. 1(b). The results in Fig. 1(b) are qualitatively similar to those of SCSs in Fig. 1(a) for the same average particle numbers, while interestingly Gaussian states are more robust against decoherence for large average particle numbers.

Our results with Gaussian states may not be very clear at first glance since Gaussuan states do not show visible interference patterns nor negativity in the Wigner function. From our viewpoint, one reason for these reasonable results is that shrinking of the Wigner function into a narrow region in phase space causes large frequency components to be dominant, which is a well-known characteristic of the Fourier transform.

Recently, an exceptional type of macroscopic superposition was introduced with a more realistic analogy of Schrödinger's cat paradox [22]. An example of such a state is $\propto \int d^2\alpha \mathcal{P}(V, d)\{|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle\alpha| + |\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle-\alpha|\}$ where $\mathcal{P}(V, d) = \exp[-\frac{2|\alpha - d|^2}{V-1}]$, V is the variance of a thermal mixed-state component and d the distance between those components. Such a state has prominent quantum properties but with large mixedness [22]. Our measure also sensibly quantifies such a peculiar type of macroscopic superpositions as

$$\mathcal{I}(V, d) = \mathcal{M}^2\left[e^{-S}\left(Q - \frac{S}{V^2}\right) + Q + S - \frac{8e^{-(V^2S/U)}R\{RU - 4d^2(V+1)\}}{U^3}\right] \tag{8}$$

where $\mathcal{M} = (2 + 2V^{-1}e^{-(S/2)})^{-1}$, $Q = (R/V)^2$, $R = V - 1$, $S = 4d^2/V$ and $U = V^2 + 1$. Here, $\mathcal{I}(\rho)$ can be made arbitrarily large by increasing d regardless of how large V is. The measure $\mathcal{I}(\rho)$ generally decreases by increasing V when $d \gg 0$. However, when $d = 0$, $\mathcal{I}(\rho)$ increases and saturates to a nonzero constant as $\mathcal{I}(\rho) \to 0.5$ for $V \to \infty$, for which "nonclassicality" is effectively evidenced even though the Wigner function has neither a negative part nor squeezing properties. All these results with Eq. (8) are

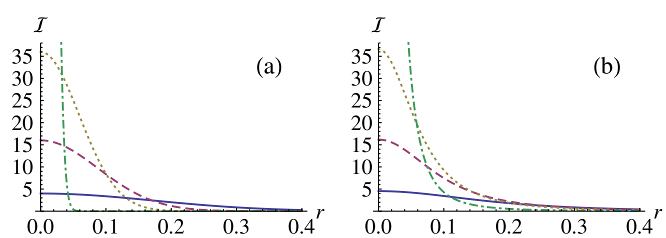

FIG. 1 (color online). (a) Interference-based measure $\mathcal{I}(\rho)$ for SCSs of amplitudes $\alpha = 2$ (solid line), $\alpha = 4$ (dashed line), $\alpha = 6$ (dotted line) and $\alpha = 27.3$ (dash-dotted line) against the normalized time r under the decoherence effect. The average number of particles is $\langle \hat{n} \rangle \approx \alpha^2$. (b) $\mathcal{I}(\rho)$ for single-mode Gaussian states of squeezing parameters $s = 1.5$ (solid line), $s = 2.1$ (dashed line), $s = 2.5$ (dotted line) and $s = 7$ (dash-dotted line), where $\langle \hat{n} \rangle = \sinh^2 r$. The same curve types mean (nearly) the same average particle numbers.

perfectly in agreement with the tendency of the Bell inequality violations closely investigated with this type of state in Ref. [22].

Since the Fourier transform is invariant of any translation and rotation in the integration region, $\mathcal{I}(\rho)$ is also invariant of any translation and rotation in phase space, i.e., $\mathcal{I}(U\rho U^\dagger) = \mathcal{I}(\rho)$, where U is such a translation or rotation. This property frustrates certain attempts to "artificially" increase $\mathcal{I}(\rho)$ by adding particles to the systems. For example, a coherent state $|\alpha\rangle$ displaced from the vacuum state $|0\rangle$ has the same value of $\mathcal{I} = 0$ as $|0\rangle$. It is worth noting that having no preferred basis states is implied as one of the necessary conditions of a measure being a faithful size criterion for macroscopic superpositions in Ref. [12] and it seems that our measure satisfies such requirement.

We point out that $\mathcal{I}(\rho)$ does not need any asymptotic assumptions or optimization techniques as in Refs. [13,15]. From a practical point of view, it is very simple to calculate using any of Eqs. (1)–(3) for an arbitrary quantum state. Especially, even for an experimentally generated state, $\mathcal{I}(\rho)$ can be evaluated using definition (2) based on the Wigner function reconstructed by the tomography technique, i.e., without the help of the fidelity with respect to a target state, its quality can readily be assessed.

To extend our proposal to atomic or spin systems represented in a finite-dimensional Hilbert space, one needs to apply the discrete Wigner function and Fourier transform. In this case, the corresponding master equation is also replaced with an appropriate one considering the underlying Hilbert space. We finally note that our measure does not suggest a threshold beyond which a superposition is "macroscopic" but rather it provides a continuous scale to compare sizes of different superpositions.

In summary, we have proposed a measure to quantify macroscopic quantum superpositions. Using our measure, true quantum coherence and the effective size of the system that involves the superposition are simultaneously quantified. Interestingly, it is directly connected to a well-known decoherence model and corresponds to the decay rate of the purity for the given state. It has been found from this relevance that our general measure is in accordance with Dür *et al.*'s designed for a specific type of states [11]. Since $\mathcal{I}(\rho)$ is based on the Wigner representation, which completely describes a quantum state, it is decomposition independent and easy to calculate for any states represented in phase space including mixed states, giving definite values for direct comparison between different types of states. All these features are hardly seen in previously proposed measures. Our measure will be widely useful for theoretical and experimental studies on macroscopic quantum systems and various related issues.

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[19] The following proof is for the single-mode case but simply extended to multimodes. When evaluating $\mathrm{Tr}(\rho\sigma)$ for positive matrices ρ and σ, one can assume without loss of generality that ρ is a diagonal matrix since ρ can be diagonalized as $\rho = U^\dagger\tilde{\rho}U$, $\mathrm{Tr}(\rho\sigma) = \mathrm{Tr}(\tilde{\rho}U\sigma U^\dagger)$ and then $U\sigma U^\dagger$ is also a positive matrix. Therefore, since the diagonal elements $\sigma_{ii} \geq 0$ for a positive matrix σ, we have $\mathrm{Tr}(\rho\sigma) = \sum_i \rho_{ii}\sigma_{ii} \leq \rho_{ii}^{\max}\sum_i\sigma_{ii} \leq \mathrm{Tr}(\sigma)$, where the equality sign holds when $\rho_{ii}^{\max} = 1$, that is, ρ is a pure state. Substituting $\sigma = (\rho\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\rho)/2$ in the above relation, we get $\mathrm{Tr}[\rho(\rho\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a}\rho)/2] \leq \mathrm{Tr}(\rho\hat{a}^\dagger\hat{a}) = \langle\hat{n}\rangle$. Similarly, since $\mathrm{Tr}(\rho\sigma) \geq \rho_{ii}^{\min}\sum_i\sigma_{ii} \geq 0\cdot\mathrm{Tr}(\sigma) = 0$, we also get $\mathrm{Tr}(\rho\hat{a}\rho\hat{a}^\dagger) \geq 0$, and finally $\mathrm{Tr}[\rho\mathcal{L}(\rho)] \leq \langle\hat{n}\rangle - 0 = \langle\hat{n}\rangle$, where the equality sign holds if and only if ρ is a pure state and $\mathrm{Tr}(\rho\hat{a}\rho\hat{a}^\dagger) = 0$.
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