Experimental demonstration of error-insensitive approximate universal-NOT gates

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We propose and experimentally demonstrate an approximate universal-NOT (UNOT) operation that is robust against operational errors. In our proposal, the UNOT operation is composed of stochastic unitary operations represented by the vertices of regular polyhedrons. The operation is designed to be robust against random operational errors by increasing the number of unitary operations (i.e., reference axes). Remarkably, no increase in the total number of measurements nor additional resources are required to perform the UNOT operation. Our method can be applied in general to reduce operational errors to an arbitrary degree of precision when approximating any antiunitary operation in a stochastic manner.

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I. INTRODUCTION

For the implementation of quantum information processing, it is necessary to reduce errors and their effects on whole processes. Quantum information processing such as quantum computing and communications is composed of three stages: state preparation, operations, and measurements. Any physical process that is described, either implicitly or explicitly, within the framework of quantum mechanics consists of these three stages and errors may occur during any processes. Errors are detrimental to the final outcome in the measurement stage: For example, both the decoherence process that changes the state before the final measurement and the inefficiency of the measurement device (i.e., detector) will affect the results. Recently, it was reported that the inaccuracy of unitary operations such as incorrect changes in the references for results to the input state or operational errors. In our proposal, the UNOT operation is composed of stochastic unitary operations represented by the vertices of regular polyhedrons. The operation is designed to be robust against random operational errors by increasing the number of unitary operations (i.e., reference axes). Remarkably, no increase in the total number of measurements nor additional resources are required to perform the UNOT operation. Our method can be applied in general to reduce operational errors to an arbitrary degree of precision when approximating any antiunitary operation in a stochastic manner.

II. CONCEPT AND THEORY

The UNOT gate is represented by the mapping $|\psi\rangle \mapsto |\psi_\perp\rangle$, where $|\psi\rangle$ is an arbitrary unknown input state in a qubit and $|\psi_\perp\rangle$ is its orthogonal state. It is well known that such a gate cannot be completely realized, but only approximately implemented [6]. To evaluate the approximate UNOT gate, we introduce two measures, the average fidelity $F$ and fidelity $\mathcal{F}$.
deviation $\Delta$, defined as
\[
F = \int f(\psi)d\psi, \quad \Delta = \sqrt{\int f(\psi)^2d\psi - F^2},
\]
where $f(\psi)$ is the fidelity between the orthogonal state and the output state of the approximate operation $O$ for a pure input state $|\psi\rangle$, i.e., $f(\psi) = \langle\psi|O(\psi)|\psi\rangle$. Note that $F$ can be maximized to $\frac{2}{3}$ (the so-called optimality condition) and $\Delta$ can be 0 (the so-called universality condition); these are regarded as the best conditions for realizing an optimum approximate UNOT gate.

In Ref. [14], it was demonstrated that the approximate UNOT gate can be realized with three or more stochastic unitary operations such that $\rho \mapsto O(\rho) = \sum_{i=1}^{m} p_i U_i \rho U_i^\dagger (m \geq 3)$, where $U_i$ is a single-qubit unitary operation given by $U_i = \cos \frac{\theta_i}{2} I + \sin \frac{\theta_i}{2} (\hat{\sigma} \cdot \vec{n}_i)$. Here $\vec{n}_i = (n_{i1}, n_{i2}, n_{i3})$ is a normalized (real) vector, i.e., $|\vec{n}_i| = 1$, and $\{p_i\}$ is the probability distribution of the stochastic operations such that $\sum p_i = 1$. The necessary condition for obtaining the maximum $F$ is that the rotation angles $\{\theta_i\}$ are $\pi$ (for all $i = 1, 2, \ldots, m$). However, perfect universality (i.e., $\Delta = 0$) is achieved by choosing an appropriate set of normalized directional vectors $\{\vec{n}_i\}$ and a suitable probability distribution $\{p_i\}$. We find conditions of $\{\vec{n}_i\}$ for a uniform distribution of $\{p_i\}$: $\{\vec{n}_i\}$ point to the vertices of regular polyhedrons [17] that are equally distributed in a solid angle. Based on the above descriptions, our generalized stochastic process for the operation is as follows:
\[
\rho \mapsto \rho_N = O_N(\rho) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\sigma} \cdot \vec{n}_i) \rho (\hat{\sigma} \cdot \vec{n}_i).
\]

For the cases of $O_3$ and $O_4$, the $\{\vec{n}_i\}$ are given by $\{(1,0,0),(0,1,0),(0,0,1)\}$ and $\{(\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}, 0), (\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{1}{3}}), (0, \sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}), (\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}})\}$, which correspond to the vertices of an octahedron and a tetrahedron, respectively. We can easily generalize these to the cases of $O_5$ and $O_6$ by considering the opposite directional vectors $\{-\vec{n}_i\}$. If error-free (in the ideal case), all the maps $O_N$ in Eq. (2) are equivalent to that of $N = 3$, the Hillery-Bužek UNOT gate [18,19] as $\rho \mapsto O_3(\rho) = \frac{1}{2} (\sigma_0 \rho \sigma_0 + \sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2)$. Thus, Eq. (2) is the optimum approximate UNOT gate.

We now consider the maps $O_N$ with errors by taking realistic circumstances into account. Errors usually deteriorate the average fidelity and fidelity deviation in implementations of the approximate UNOT gate and here we consider a specific but very common (operational) error that arises from the imperfect setting of $U_i$. It is important to note that $\Delta$ can be seriously affected by even a small error, whereas $F$ will remain close to its maximum value of $\frac{2}{3}$ [14]. This trend motivated us to invent an error-insensitive approximate UNOT gate, significantly reducing the influence of the errors on $\Delta$.

We show that adding more stochastic operations will increase the resilience against the operational errors. This can be verified by analytic calculations for the cases of $N = 3, 4, 6,$ and 8. The process in Eq. (2) is characterized by $\chi$ matrices through $O(\rho) = \sum_{I,j=0}^{3} \chi_{Ij} \sigma_I \rho \sigma_j$, where $\sigma_0 = I$ [4], and so the ideal case of the approximate UNOT gate is characterized by $\chi_I = \text{diag}(0,1,1,1)$. However, operational errors occurring under realistic circumstances will vary $\chi_I$. By using the $\chi$ matrix, we can find the mean of average fidelity $\bar{F}_N$ and fidelity deviation $\bar{\Delta}_N$ for the map $O_N$ over the random errors as
\[
\bar{F}_N \simeq \frac{2}{3}, \quad \bar{\Delta}_N \simeq \frac{\alpha}{\sqrt{N}} \delta_N = \frac{S_N \delta}{\sqrt{N}},
\]
where $\alpha$ is a constant factor and $\delta_N$ is the standard deviation of the random error. Since $\bar{F}_N$ is stationary, we define the error sensitivity of $O_N$ with $\bar{\Delta}_N$ as $S_N \equiv \alpha / \sqrt{N}$ (see Appendix C for more details). In deriving Eq. (3), the random error is considered as a random unitary operation $V_i$ following the stochastic operation, e.g., $\hat{\sigma} \cdot \vec{n}_i \rightarrow V_i (\hat{\sigma} \cdot \vec{n}_i)$. We consider the error operation as $V_i = e^{i \vec{\epsilon}_i} \hat{\sigma} \rightarrow 1 + i \vec{\epsilon}_i \cdot \hat{\sigma}$, where $\vec{\epsilon}_i = (\epsilon_{i1}, \epsilon_{i2}, \epsilon_{i3})$ and $|\epsilon_{ij}| < \epsilon_0 \ll 1$, so that the error distribution is symmetric and homogeneous (see Appendix B for more details). Equation (3) shows directly that the deterioration of $\Delta_N$ due to operational errors can be reduced by simply increasing the number of stochastic operations $N$.

Note that $N$ is the number of stochastic operations that constitute the map $O_N$ and is not the number of measurements. However, we see that Eq. (3) is very similar in form to that for the standard error of $N$ measurements of random variables. The reason for this similarity is that the output state of the map is a mixed state: a convex combination of $N$ states. An erroneous output state of $O_N$ is expressed as a summation of vectors in the Bloch sphere as $\sum_{i=1}^{N} (\vec{\epsilon}_i + \vec{\omega}_i)/N$, where $\vec{\omega}_i$ are the Bloch vectors of ideal stochastic operations and $\vec{\epsilon}_i$ are effects of the operational errors [20]. The total effect of operational errors is described as $\sum_{i=1}^{N} \vec{\epsilon}_i /N$ with an average of zero and a standard deviation proportional to $\chi(\vec{\epsilon}_i) / \sqrt{N}$. Therefore, the average fidelity remains close to the maximum and a mean of $\Delta_N$ for an erroneous map is expressed by Eq. (3).

### III. SETUP AND METHOD

The experiment was based on the polarization state of a single photon generated from SPDC and manipulated by linear...
optics, as shown in Fig. 1. We first generated a pair of photons via a frequency-degenerate collinear type-II SPDC process using a diode continuous-wave laser (1.8 mW at 406 nm) and a periodically poled KTiOPO4 crystal ($L = 10$ mm, $\Lambda = 10.00$ $\mu$m). The polarizations of the two photons in a single mode (the same frequency and spatial mode) were orthogonal to each other as $|HV\rangle$. A horizontal photon transmitting the polarizing beam splitter (PBS) was controlled to be in an arbitrary polarization qubit state through the use of the half-wave plate (HWP) and the quarter-wave plate (QWP). The vertical photon plays the role of a counting trigger for coincidence counts.

An arbitrary unitary operation for the polarization qubit can be realized by a set of wave plates (QWP-HWP-QWP) [21]. We stochastically perform unitary operations $\{\vec{\sigma} \cdot \vec{n}_i\}$ to realize the map $O_N$. The random error of a unitary operation is achieved by rotating each wave plate in the set randomly between $[-\phi_e, \phi_e]$. The output states are measured by a polarization analyzer (QWP-HWP-PBS) and reconstructed by quantum state tomography (QST) [22]. To obtain $F_N$, $\Delta_N$, and $S_N$ of each stochastic map $O_N$, we execute QPT, which characterizes a quantum operation by means of the QST results for four input states and their output states. From the result of QPT and $\chi$ matrix, we can calculate $F_N$ and $\Delta_N$ (see Appendix A for more details). To survey $S_N$, we repeat the QPT measurements of $O_N$ for $N = 3, 4, 6, 8$ by varying the boundaries of the random error $\phi_e$ from $0^\circ$ to $5^\circ$ [23].

IV. RESULTS AND ANALYSIS

Figure 2 shows reconstructed input states and their output states for QPTs of the maps $O_{3,4,6,8}$ in the Bloch spheres [24] and $\chi_N$ matrices when there are no operational errors. The points (in black) on the surface are the input states and the points (in red, green, blue, and orange) close to the center represent the output states of the maps $O_{3,4,6,8}$ [25]. The graphs show clearly that the output states are on opposite sides of the input states and that their lengths decrease by about $1/3$. The $\chi_N$ matrices are calculated from the reconstructed density matrices of the input and output states and these are almost the same as those of the ideal case, i.e., $\chi_{11} = \chi_{22} = \chi_{33} = \frac{1}{3}$. This constitutes experimental verification of the equivalence of the maps $O_N$ for $N = 3, 4, 6, 8$ under error-free circumstances.

Figure 3 represents the input and output states of erroneous maps $O_{3,4,6,8}^{\text{err}}$ for $\phi_e = 5^\circ$ [24]. The output states of $O_N^{\text{err}}$ are more diffused than those of $O_N$, where $N < M$, and we see that for larger $N$, the map $O_N$ is less sensitive to random operational errors. Since the output states have a distribution in the Bloch sphere, we describe the erroneous maps $O_N^{\text{err}}$ as the average value of $F_N$ and $\Delta_N$ over 20 QPT results.

We repeated the QPT experiments for the maps $O_{3,4,6,8}^{\text{err}}$ for various error boundaries $\phi_e$ to measure $S_N$. The average values of $F_N$ and $\Delta_N$ for random errors are shown in Fig. 4. Each point represents the average value of 20 experimental results, except for the error-free case ($\phi_e = 0^\circ$) in which two results were averaged. The solid and dashed lines represent linear fits.
of the experimental data and trends of the simulation results for \(10^4\) QPT measurements, respectively. The simulation results are shifted up about 0.005 to match the experimental results. We attribute this offset to inevitable imperfections in the experimental setup and measurements.

In Fig. 4 the results of \(\Delta N\) are distinct for different \(N\), although there are no significant differences in \(\overline{F}_N\). The experimental data of \(\Delta N\) and \(\overline{F}_N\) agree well with the simulation results to within the deviations. From Eq. (3) we note that the gradients of the \(\Delta N\) curves corresponding to the sensitivity \(S_N\) and the relative gradients \(S_N/S_M\) obtained from the experimental data (simulation results) for \(\Delta N\) and \(\overline{F}_M\) obey the relation \(\sqrt{M/N}\) to within an accuracy of about 82% (99%). This shows that the sensitivity of the stochastic map \(O_N\) is inversely proportional to the square root of the number of operations \(N\) as described in Eq. (3).

Note that all the experimental data (one QST or one QPT) were measured using the same amount of resources (photon pairs). Thus, the error insensitivity of \(O_N\) is not a matter of the measurement precision dependent on the number of repetitions. The essential point of our scheme is that the output state is a convex combination of other states, so we expect that our error-insensitive method will be applicable to other stochastic operations. We utilized the stochastic method rather than the ancilla-assisted model, which demands \(N−1\) controlled unitary operations for \(O_N\), so as to avoid any controlled unitary operations that are probabilistic in the framework of linear optics. Most cases of the ancilla-assisted model are composed of sequential controlled unitary operations so that their operational errors can be accumulated. Therefore, to apply our method to the ancilla-assisted model, a specific controlled operation and an equally superposed ancilla state are required, e.g., \(U(\pi)_{AB} = \sum_{i=1}^{N} |i\rangle_A \langle i| \otimes (\vec{\sigma} \cdot \vec{n}_i)_B\) and \(|\psi_0\rangle_A = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle\), where \(A\) and \(B\) denote ancilla and object systems, respectively.

V. CONCLUSION

We have introduced an error-insensitive (robust) UNOT gate consisting of stochastic unitary operations with rotation axes corresponding to the vertex directions of an octahedron and hexahedron and a rotation angle of \(\pi\). We demonstrated both theoretically and experimentally that the sensitivity of the map to random operational errors is inversely proportional to the square root of the number of stochastic operations. The method does not require any increase in the total number of measurements nor additional resources. Even though we have considered only the maps \(O_N\) for \(N = 3, 4, 6,\) and 8, our scheme can be generalized to \(N = 3n\) and \(4n\) (\(1 \leq n\)). This method is also applicable to all approximate antiunitary operations, since such operations are equivalent to a unitary transformation, and it may be possible to extend the method to other stochastic mappings.

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APPENDIX A: CALCULATION OF $F$ AND $\Delta$
FROM THE $\chi$ MATRIX

When a quantum operation $O$ is characterized by a $\chi$ matrix as
\[
\chi = \begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} \\
X_{15} & X_{22} & X_{23} & X_{24} \\
X_{13} & X_{32} & X_{33} & X_{34} \\
X_{14} & X_{24} & X_{34} & X_{44}
\end{pmatrix},
\] (A1)

the output state is expressed as
\[
O(\psi) = \sum_{i,j=0}^{3} \chi_{ij} \sigma_i |\psi\rangle \langle \psi | \sigma_j,
\] (A2)

where $\sigma_0$ is $2 \times 2$ identity matrix and $\sigma_i \neq 0$ are the Pauli matrices. The fidelity between the output state of the map $O$ and ideal UNOT gate is described as
\[
f(O(\psi), \psi_\perp) = \langle \psi_\perp | O(\psi) | \psi_\perp \rangle
\]
\[
= \sum_{i,j=0}^{3} \chi_{ij} \sigma_i |\psi\rangle \langle \psi | \sigma_j |\psi\rangle
\]
\[
= \sum_{i,j=0}^{3} \chi_{ij} C_i C_j^*,
\] (A3)

where the coefficients are defined as $C_i \equiv |\psi_\perp | \sigma_i |\psi\rangle$. Since an arbitrary pure state and its orthogonal state can be represented by $|\psi\rangle = \cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle$ and $|\psi_\perp\rangle = \sin \frac{\theta}{2} |0\rangle - e^{i\phi} \cos \frac{\theta}{2} |1\rangle$, respectively, where $\theta = [0, \pi]$ and $\phi = [0, 2\pi]$, the coefficients $C_i$ are functions of $\theta$ and $\phi$. Thus, the average fidelity $F$ and the square of the fidelity deviation $\Delta^2$ are obtained as
\[
F = \frac{1}{4\pi} \sum_{i,j=0}^{3} \chi_{ij} \int C_i C_j^* \sin \theta \, d\theta \, d\phi
\]
\[
= \frac{2}{3} (\chi_{11} + \chi_{22} + \chi_{33}),
\] (A4)
\[
\Delta^2 = \frac{1}{4\pi} \sum_{i,j,k,l=0}^{3} \chi_{ij} \chi_{kl} \int C_i C_j^* C_k C_l^* \sin \theta \, d\theta \, d\phi - F^2
\]
\[
= \frac{4}{45} (\chi_{11}^2 + \chi_{22}^2 + \chi_{33}^2 - \chi_{11} \chi_{22} - \chi_{11} \chi_{33} - \chi_{22} \chi_{33})
\]
\[
+ \frac{4}{15} (3|\chi_{12}|^2 + 3|\chi_{13}|^2 + 3|\chi_{23}|^2
\]
\[
- 2 \text{Re}[\chi_{12}^2 + \chi_{13}^2 + \chi_{23}^2]).
\] (A5)

APPENDIX B: RANDOM OPERATIONAL ERROR
AND $O^err_N$

A random operational error can be considered as an additional random unitary operation following the original operation $\tilde{\sigma} \cdot \tilde{n}_i \longrightarrow V_i (\tilde{\sigma} \cdot \tilde{n}_i)$. The error operation is defined as $V_i = e^{i\tilde{\epsilon}_i \cdot \tilde{\sigma}} \simeq 1 + i \tilde{\epsilon}_i \cdot \tilde{\sigma}$, where $\tilde{\epsilon}_i = (\epsilon_{i1}, \epsilon_{i2}, \epsilon_{i3})$ and $|\epsilon_{ij}| \leq e_0 \ll 1$. The distribution of errors $P(\tilde{\epsilon}_i)$ is assumed to be homogeneous and symmetric under the inversion $P(-\tilde{\epsilon}_i) = P(\tilde{\epsilon}_i)$ so that $\tilde{\epsilon}_i = 0$. Then, the erroneous map $O_N^{err}$ is expressed as
\[
\rho \rightarrow \rho_N^{err} = O_N^{err} (\rho) = \frac{1}{N} \sum_{i=1}^{N} V_i (\tilde{\sigma} \cdot \tilde{n}_i) \rho (\tilde{\sigma} \cdot \tilde{n}_i) V_i^\dagger.
\] (B1)

APPENDIX C: PROOF OF EQ. (3)

1. The case of $O_3^{err}$

The first order of error terms $\delta O_3^{(1)}$ for the map $O_3^{err}$ and their additional contribution $\delta \chi_3^{(1)}$ to $\chi_3$ matrix are
\[
\delta O_3^{(1)} = \frac{1}{3} \left[ i (\tilde{\epsilon}_1 \cdot \tilde{\sigma}) \sigma_x \rho \sigma_x + i (\tilde{\epsilon}_2 \cdot \tilde{\sigma}) \sigma_y \rho \sigma_y + i (\tilde{\epsilon}_3 \cdot \tilde{\sigma}) \sigma_z \rho \sigma_z + c.c. \right],
\] (C1)
\[
\delta \chi_3^{(1)} = \frac{1}{3} \begin{pmatrix}
0 & i\epsilon_{21} & 0 & i\epsilon_{31} \\
-i\epsilon_{12} & 0 & \epsilon_{13} + \epsilon_{23} & 0 \\
-i\epsilon_{23} & \epsilon_{23} - \epsilon_{13} & 0 & \epsilon_{31} \\
-i\epsilon_{31} & -\epsilon_{32} & \epsilon_{31} - \epsilon_{21} & 0
\end{pmatrix}.
\] (C2)

Using Eqs. (A4), (A5), and (C2), the average fidelity and the fidelity deviation for $O_N^{err}$ are obtained as
\[
F_3 = \frac{2}{3},
\] (C3)
\[
\Delta_3 = \frac{2\sqrt{(\epsilon_{13} - \epsilon_{23})^2 + (\epsilon_{21} - \epsilon_{31})^2 + (\epsilon_{12} - \epsilon_{32})^2}}{3\sqrt{15}}
\]
\[
= \frac{2\sqrt{2}}{3\sqrt{15}} \sqrt{r_1^2 + r_2^2 + r_3^2},
\] (C4)

where $|r_i|$ are replaced random variables of $|\epsilon_{ij}|$ up to normalization factor $\sqrt{2}$. Note that the errors of which directions are parallel to the original operations, i.e., $|\epsilon_{ij}|$, do not contribute to the average fidelity and the fidelity deviation as shown in Eqs. (C2)–(C4). A mean of the fidelity deviation over random errors is proportional to the standard deviation of random variables,
\[
\overline{\Delta}_3 = \sqrt{\frac{8}{15} \delta r},
\] (C5)

where $\delta r = \overline{r_i^{1/2}}$ and we assume $r_i = 0$.

2. The case of $O_4^{err}$

After tedious calculations, the first order of error terms $\delta O_4^{(1)}$ and their contribution $\delta \chi_4^{(1)}$ to the $\chi_4$ matrix are expressed as
\[
\delta O_4^{(1)} = -\frac{1}{12} \left[ (\epsilon_{11} \sqrt{3} \sigma_x + \epsilon_{12} (\sigma_x - \sqrt{2} \sigma_y)) \rho (\sqrt{2} \sigma_x + \sigma_y) + (\epsilon_{21} \sqrt{3} \sigma_x + \epsilon_{22} (\sigma_x + \sqrt{2} \sigma_y)) \rho (-\sqrt{2} \sigma_x + \sigma_y)
\]
\[
+ (\epsilon_{31} \sqrt{3} \sigma_x + \epsilon_{32} (\sqrt{2} \sigma_x + \sigma_y)) \rho (\sigma_x + \sqrt{2} \sigma_x) + (\epsilon_{41} \sqrt{3} \sigma_x + \epsilon_{42} (\sqrt{2} \sigma_x + \sigma_y)) \rho (-\sigma_x - \sqrt{2} \sigma_x) + c.c. \right],
\] (C6)
\[\delta \chi_4^{(1)} = \frac{1}{12} \begin{pmatrix} 0 & 0 & \epsilon_{12} + \epsilon_{22} + \sqrt{3}(\epsilon_{31} + \epsilon_{41}) \\ 0 & 2\sqrt{3}(\epsilon_{12} + \epsilon_{22}) & \epsilon_{12} + \epsilon_{22} + \sqrt{3}(\epsilon_{31} + \epsilon_{41}) \\ 2\sqrt{3}(\epsilon_{12} + \epsilon_{22}) & 0 & -\sqrt{3}(\epsilon_{11} + \epsilon_{21}) - \epsilon_{32} - \epsilon_{42} \end{pmatrix} \].

(C7)

In this case, for a simple calculation, we neglect errors that are parallel to the original operations since the errors do not contribute to the first-order calculation as shown in the case of \( O_4^e \). Using Eqs. (A4), (A5), (C7), and replaced random variables, \( F_4 \) and \( \Delta_4 \) for \( O_4^e \) are obtained as

\[ F_4 = \frac{2}{3}, \]

\[ \Delta_4 = \frac{1}{3\sqrt{30}} \sqrt{(3\alpha_+ + \delta_+)^2 + 6(\alpha_- + \gamma_-)^2 + (\beta_+ + \sqrt{3}\gamma_+)^2 + 8(\beta_-^2 + \beta_-\delta_- + \delta_-^2)} \]

\[ = \frac{\sqrt{2}}{3\sqrt{15}} \sqrt{R_1^2 + 3R_2^2 + 3R_3^2 + R_4^2}, \]

(C9)

where the replaced variables are defined as

\[ \alpha_\pm = \frac{\epsilon_{11} \pm \epsilon_{31}}{\sqrt{2}}, \quad \beta_\pm = \frac{\epsilon_{12} \pm \epsilon_{22}}{\sqrt{2}}, \]

\[ \gamma_\pm = \frac{\epsilon_{31} \pm \epsilon_{41}}{\sqrt{2}}, \quad \delta_\pm = \frac{\epsilon_{32} \pm \epsilon_{42}}{\sqrt{2}}, \]

\[ R_1 = \frac{3\alpha_+ + \delta_+}{2}, \quad R_2 = \frac{\alpha_- + \gamma_-}{\sqrt{2}}, \quad R_3 = \frac{\beta_- + \sqrt{3}\gamma_+}{2}, \]

\[ R_4 = \frac{\beta_- - \delta_-}{\sqrt{2}}. \]

(C10)

A mean of \( \Delta_4 \) over random errors is obtained as

\[ \overline{\Delta_4} = \sqrt{\frac{8}{15} \delta_r / 4}. \]

(C12)

### 3. The cases of \( O_4^e \) and \( O_4^e \)

From tedious calculations, average fidelities are the same as \( \frac{2}{3} \) and means of \( \Delta_6 \) and \( \Delta_8 \) over random errors are obtained as

\[ \overline{\Delta_6} = \sqrt{\frac{8}{15} \delta_r / 6}, \]

\[ \overline{\Delta_8} = \sqrt{\frac{8}{15} \delta_r / 8}. \]

(C13)

From Eqs. (C5) and (C12)–(C14), we infer that the average of \( \Delta_N \) (at least for the cases of \( N = 3n, 4n \)) over random errors is expressed as

\[ \overline{\Delta_N} = \sqrt{\frac{8}{15} \delta_r / N}. \]

(C15)

### References


[17] The number of vertices for tetra-, hexa-, octa-, dodeca-, and icosahedrons are 4, 8, 6, 20, and 12, respectively. However, half of the points for each polyhedron are located on opposite sides of the other half, except the tetrahedron. Thus, the minimum number of stochastic operations to satisfy the universality condition, represented by each polyhedron, is 4, 4, 3, 10, and 6, respectively. The cases of tetra- and hexahedrons are equivalent.


[20] The \( \hat{e}_i \) has symmetric and random distributions on an orthogonal plane to \( \hat{e}_i \) when \( |\hat{e}_i| \ll 1 \).


[23] Although experimental uncertainty of the optical axis is less than 2° in general, we examine up to the case of \( \phi_e = 5° \) to clearly
show the $S_N$. The standard deviation of random errors between $[-\phi_e, \phi_e]$ is proportional to $\phi_e$.


[25] We repeated QSTs of all the input and output states ten times to analyze the effects of photon counting fluctuations. However, the distribution of reconstructed states is less than the size of a point on the Bloch sphere. Thus we can neglect photon counting fluctuation errors.