Witnessing entanglement in phase space using inefficient detectors

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We propose a scheme for witnessing entanglement in phase space by significantly inefficient detectors. The implementation of this scheme does not require any additional process for correcting errors, in contrast to previous proposals. Moreover, it allows us to detect entanglement without full *a priori* knowledge of the detection efficiency. It is shown that entanglement in single-photon entangled and two-mode squeezed states can be witnessed with detection efficiency as low as 40%. Our approach enhances the possibility of witnessing entanglement in various physical systems using current detection technologies.

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I. INTRODUCTION

Entanglement is one of the most remarkable features of quantum mechanics which cannot be understood in the context of classical physics. It has been shown that entanglement can exist in various physical systems [1,2] and play a role in quantum phenomena such as quantum phase transition [3]. Moreover, its properties can be used as a resource for quantum information technologies such as quantum computing, quantum cryptography, and quantum communication [4]. Therefore, detecting entanglement is one of the most essential tasks both for studying fundamental quantum properties and for applications in quantum information processing. Although various entanglement detection schemes have been proposed [5,6], their experimental realization suffers from imperfections of realistic detectors since measurement errors wash out quantum correlations. This difficulty becomes more significant with increasing system dimensionality and particularly in continuous variable systems where entanglement is increasingly attracting interest [7].

Quantum tomography provides a method to reconstruct complete information of quantum states in phase-space formalism [8–10]. The reconstructed data can be used to determine whether the state is entangled or not with the help of an entanglement witness (EW) [6]. Bell inequalities that were originally derived for discriminating quantum mechanics from local realism [11] can also be used for witnessing entanglement since their violation guarantees the existence of entanglement. Banaszek and Wódkiewicz (BW) [12] suggested a Bell-type inequality (referred to as BW inequality in this article) which can be tested by way of reconstructing the Wigner function at a few specific points of phase space. However, imperfections of tomographic measurements constitute a crucial obstacle for practical applications. Several schemes have been considered to overcome this problem [10], such as numerical inversion [13] and maximum-likelihood estimation [14], but they require a great amount of calculations or iteration steps for high-dimensional and continuous variable systems.

In this article we propose an alternative entanglement detection scheme in phase-space formalism, which can be used in the presence of detection noise. We formulate an EW in the form of a Bell-like inequality using the experimentally measured Wigner function. For this, we include the effects of detection efficiency into possible measurement outcomes. Possible expectation values of the EW are bounded by the maximal expectation value when separable states are assumed. Any larger expectation value guarantees the existence of entanglement.

Our approach shows the following remarkable features: (i) in contrast to previous proposals [10,14], it does not require any additional process for correcting measurement errors; (ii) it allows us to witness entanglement, for example, in single-photon entangled and two-mode squeezed states with detection efficiency as low as 40%; (iii) our scheme is also valid when the precise detection efficiency is not known prior to the test; (iv) finally, we note that our approach is applicable for detecting any quantum state represented in phase-space formalism.

We will first define an observable operator associated with the detection efficiency and the experimentally measured Wigner function in Sec. II. We then formulate an EW in the form of Bell-type inequality based on the phase-space formalism in Sec. III. Using the proposed EW, we demonstrate the detection of entanglement in the single-photon entangled (Sec. IV) and two-mode squeezed states (Sec. V) with inefficient detectors. We also discuss the effects of imperfect estimation of detection efficiency in Sec. VI and conclude this article in Sec. VII.

II. OBSERVABLE ASSOCIATED WITH EFFICIENCY

We begin by introducing an observable associated with the detection efficiency η and an arbitrary complex variable α:

$$\hat{O}(\alpha)=\begin{cases}\frac{1}{\eta}\hat{\Pi}(\alpha)+\left(1-\frac{1}{\eta}\right)\mathbb{1} & \text{if } \frac{1}{2}<\eta\leqslant 1,\\ 2\hat{\Pi}(\alpha)-\mathbb{1} & \text{if } \eta\leqslant\frac{1}{2},\end{cases}\tag{1}$$

where $\hat{\Pi}(\alpha)=\sum_{n=0}^{\infty}(-1)^n|\alpha,n\rangle\langle\alpha,n|$ is the displaced parity operator and $\mathbb{1}$ is the identity operator. $|\alpha,n\rangle=\hat{D}(\alpha)|n\rangle$ is the displaced number state produced by applying the Glauber displacement operator $\hat{D}(\alpha)$ to the number state $|n\rangle$.

Let us then consider the expectation value of observable (1) when the measurement is carried out with efficiency η. In general, measurement errors occur when not all particles are counted in the detector. Thus, the real probability distribution of particles, $P(n)$, transforms to another distribution, $P_\eta(m)$, by the generalized Bernoulli transformation [15]:

　

$P_\eta(m) = \sum_{n=m}^{\infty} P(n)\binom{n}{m}(1-\eta)^{n-m}\eta^m$. Thus, the expectation value of the parity operator is obtained as

$$\langle\hat{\Pi}(\alpha)\rangle_\eta = \sum_{m=0}^{\infty}(-1)^m P_\eta(\alpha,m) = \sum_{n=0}^{\infty}(1-2\eta)^n P(\alpha,n), \quad (2)$$

where $\langle\cdot\rangle_\eta$ implies the statistical average measured with efficiency η. Here $P_\eta(\alpha,m)$ and $P(\alpha,n)$ are the measured and real particle number distributions in the phase space displaced by α, respectively.

We define the Wigner function experimentally measured with efficiency η as

$$W^\eta(\alpha) \equiv \frac{2}{\pi}\langle\hat{\Pi}(\alpha)\rangle_\eta, \quad (3)$$

which is given as a rescaled Wigner function by Gaussian smoothing. Note that a smoothed Wigner function can be identified with a s-parametrized quasiprobability function as $W^\eta(\alpha) = ((\alpha; -(1-\eta)/\eta))/\eta$, where $W(\alpha;s) = \{2/[\pi(1-s)]\}\sum_{n=0}^{\infty}[(s+1)/(s-1)]^n P(\alpha,n)$ [16]. This identification is available both for homodyne [8,17] and number-counting tomography methods [9]. After series of measurements with efficiency η, we can obtain the expectation value of the observable (1) as

$$\langle\hat{O}(\alpha)\rangle_\eta = \begin{cases} \frac{\pi}{2\eta}W^\eta(\alpha) + 1 - \frac{1}{\eta} & \text{if } \frac{1}{2} < \eta \leqslant 1, \\ \pi W^\eta(\alpha) - 1 & \text{if } \eta \leqslant \frac{1}{2}, \end{cases} \quad (4)$$

which is bounded as $|\langle\hat{O}(\alpha)\rangle_\eta| \leqslant 1$ for all η.

III. ENTANGLEMENT WITNESS IN PHASE SPACE

Let us formulate an EW in the framework of phase space. Suppose that two separated parties, Alice and Bob, measure one of two observables, denoted by $\hat{A}_1$, $\hat{A}_2$ for Alice and $\hat{B}_1$, $\hat{B}_2$ for Bob. All observables are variations of the operator (1) as $\hat{A}_a = \hat{O}(\alpha_a)$ and $\hat{B}_a = \hat{O}(\beta_b)$ with $a,b = 1,2$. We then formulate a Hermitian operator as a combination of each local observable $\hat{A}_a$, $\hat{B}_b$ in the form

$$\hat{\mathcal{W}} = \hat{C}_{1,1} + \hat{C}_{1,2} + \hat{C}_{2,1} - \hat{C}_{2,2}, \quad (5)$$

where $\hat{C}_{a,b} = \hat{A}_a \otimes \hat{B}_b$ is the correlation operator. We call $\hat{\mathcal{W}}$ an EW operator. Note that the operator in Eq. (5) can also be regarded as a Bell operator $\hat{\mathcal{B}}$ which distinguishes nonlocal properties from local realism. The bound expectation value of the operator in Eq. (5) is determined according to whether it is regarded as $\hat{\mathcal{W}}$ or $\hat{\mathcal{B}}$. In other words, the entanglement criterion given by the operator (5) is different from the nonlocality criterion, as we will show.

Let us first obtain the bound expectation value of the operator (5) as an EW by which one can discriminate entangled and separable states. For a separable state $\hat{\rho}_{\rm sep} = \sum_i p_i \hat{\rho}_i^A \otimes \hat{\rho}_i^B$ where $p_i \geqslant 0$ and $\sum_i p_i = 1$, the expectation value of the correlation operator measured with efficiency η is given by

$$\begin{aligned}\langle\hat{C}_{a,b}\rangle_\eta^{\rm sep} &= \sum_i p_i \sum_{n,m}^{\infty}(1-2\eta)^{n+m} \\ &\quad\times\langle\alpha,n|\hat{\rho}_i^A|\alpha,n\rangle\langle\beta,m|\hat{\rho}_i^B|\beta,m\rangle \\ &= \sum_i p_i\langle\hat{A}_a\rangle_\eta^i\langle\hat{B}_b\rangle_\eta^i. \end{aligned} \quad (6)$$

Since expectation values of all local observables with efficiency η are bounded as $|\langle\hat{A}_a\rangle_\eta^i|, |\langle\hat{B}_b\rangle_\eta^i| \leqslant 1$ for $a,b = 1,2$, we can obtain the statistical maximal bound of the EW operator (5) with respect to the separable states

$$\begin{aligned}\left|\langle\hat{\mathcal{W}}\rangle_\eta^{\rm sep}\right| &= \Big|\sum_i p_i\big(\langle\hat{A}_1\rangle_\eta^i\langle\hat{B}_1\rangle_\eta^i + \langle\hat{A}_1\rangle_\eta^i\langle\hat{B}_2\rangle_\eta^i \\ &\quad + \langle\hat{A}_2\rangle_\eta^i\langle\hat{B}_1\rangle_\eta^i - \langle\hat{A}_2\rangle_\eta^i\langle\hat{B}_2\rangle_\eta^i\big)\Big| \\ &\leqslant 2\sum_i p_i = 2 \equiv \mathcal{W}_{\max}^{\rm sep}. \end{aligned} \quad (7)$$

Therefore, if $|\langle\hat{\mathcal{W}}\rangle_\eta^\psi| > \mathcal{W}_{\max}^{\rm sep} = 2$ for a quantum state ψ, we can conclude that the quantum state ψ is entangled.

Let us then consider the operator (5) as a Bell operator. Note that the local-realistic (LR) bound of a Bell operator is given as the extremal expectation value of the Bell operator, which is associated with a deterministic configuration of all possible measurement outcomes. If $1/2 < \eta \leqslant 1$, the maximal modulus outcome of (1) is $|1 - 2/\eta|$ when the outcome of parity operator $\hat{\Pi}(\alpha)$ is measured as -1. Thus, the expectation value of (5) is bounded by local realism as $|\langle\hat{\mathcal{B}}\rangle_\eta| \leqslant \mathcal{B}_{\max}^{\rm LR} = 2(1-2/\eta)^2$. Likewise for $\eta \leqslant 1/2$, we can obtain $\mathcal{B}_{\max}^{\rm LR} = 18$. Note that $\mathcal{B}_{\max}^{\rm LR} \geqslant \mathcal{W}_{\max}^{\rm sep}$ for all η, and $\mathcal{B}_{\max}^{\rm LR} = \mathcal{W}_{\max}^{\rm sep}$ in the case of unit efficiency ($\eta = 1$). This shows that some entanglement can exist without violating local realism, and thus the Bell operator can be regarded as a nonoptimal EW as pointed out already in [18]. For the purpose of this article we will focus on the role of an EW in the following parts.

From Eqs. (5) and (7), we can finally obtain an EW in the form of an inequality obeyed by any separable state:

$$\begin{aligned}|\langle\hat{\mathcal{W}}\rangle_{\eta>\frac{1}{2}}| &= \Bigg|\frac{\pi^2}{4\eta^2}\big[W_{1,1}^\eta + W_{1,2}^\eta + W_{2,1}^\eta - W_{2,2}^\eta\big] \\ &\quad + \frac{\pi(\eta-1)}{\eta^2}\big[W_{a=1}^\eta + W_{b=1}^\eta\big] + 2\left(1-\frac{1}{\eta}\right)^2\Bigg| \leqslant 2, \\ |\langle\hat{\mathcal{W}}\rangle_{\eta\leqslant\frac{1}{2}}| &= \pi^2\big[W_{1,1}^\eta + W_{1,2}^\eta + W_{2,1}^\eta - W_{2,2}^\eta\big] \\ &\quad - 2\pi\big[W_{a=1}^\eta + W_{b=1}^\eta\big] + 2| \leqslant 2, \end{aligned} \quad (8)$$

where $W_{a,b}^\eta$ is the two-mode Wigner function measured with efficiency η (here we replace the notation α_a and β_b in the conventional representation of two-mode Wigner function $W^\eta(\alpha_a,\beta_b)$ with the notation a,b for simplicity), and $W_{a(b)=1}^\eta$ is its marginal single-mode distribution. We note that the EW in Eq. (8) can also be derived from the Bell inequality formulated using s-parametrized quasiprobability functions [19] when regarding effects of detector inefficiency as changes to s. Any violation of Eq. (8) guarantees that the measured quantum state is entangled. Remarkably, our scheme allows one to detect entanglement without correcting measurement errors. Note that in the case of unit efficiency ($\eta = 1$) the inequality in Eq. (8) becomes equivalent to the BW inequality [12]. It is also notable that any violation of this inequality for $\eta < 1$ ensures the violation of the BW inequality in the case of a unit efficiency ($\eta = 1$). Therefore, the proposed EW in Eq. (8) can be used effectively for detecting entanglement instead of the BW inequality in the presence of measurement noise.

IV. TESTING SINGLE-PHOTON ENTANGLED STATES

Let us now apply the EW in Eq. (8) for detecting entangled photons. We here first consider the single-photon entangled state $|\Psi\rangle = (|0,1\rangle + |1,0\rangle)/\sqrt{2}$, where $|0,1\rangle(|1,0\rangle)$ is the state with zero (one) photons in the mode of Alice and one (zero) photon in the mode of Bob [20]. This state can be created by a single-photon incident on a 50:50 beam splitter. Its two-mode Wigner function measured with efficiency η is

$$W^{\eta}_{a,b} = \frac{4}{\pi^2}(1 - 2\eta + 2\eta^2|\alpha_a + \beta_b|^2) \times \exp[-2\eta(|\alpha_a|^2 + |\beta_b|^2)] \quad (9)$$

and its marginal single-mode distribution is

$$W^{\eta}_{a} = \frac{1}{\pi}(2 - 2\eta + 4\eta^2|\alpha_a|^2)\exp[-2\eta|\alpha_a|^2]. \quad (10)$$

The expectation values of operator (5) with properly chosen α_a and β_b are plotted in Fig. 1(a) against the overall efficiency η. It is remarkable that entanglement can be detected even with detection efficiency η as low as 40%.

V. TESTING TWO-MODE SQUEEZED VACUUM STATES

Let us consider the EW in continuous variable systems, for example, two-mode squeezed vacuum states (TMSSs).

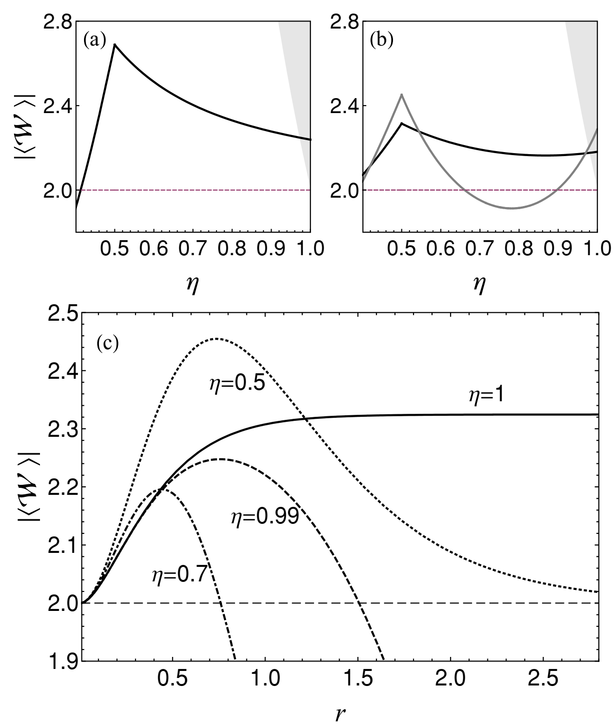

FIG. 1. (Color online) Maximum expectation value of the EW operator in Eq. (5) for an input of (a) a single-photon entangled state and (b) a two-mode squeezed state with $r = 0.4$ (black) and $r = 0.8$ (gray). Entanglement exists if the expectation value exceeds the dashed line $\mathcal{W}^{\rm sep}_{\rm max} = 2$. Note that the shaded region which exceeds $\mathcal{B}^{\rm LR}_{\rm max} = 2(1 - 2/\eta)^2$ (for $1/2 < \eta \leqslant 1$) is the criterion of nonlocality. (c) Witnessing entanglement with varying squeezing rate r of a two-mode squeezed states for detector efficiencies $\eta = 1$ (solid line), $\eta = 0.99$ (dashed line), $\eta = 0.7$ (dot-dashed line), and $\eta = 0.5$ (dotted line).

This state can be generated by nondegenerate optical parametric amplifiers [21] and be written as $|{\rm TMSS}\rangle = {\rm sech} r \sum_{n=0}^{\infty} \tanh^n r|n,n\rangle$, where $r > 0$ is the squeezing parameter. The measured Wigner function with efficiency η for a TMSS is given by

$$W^{\eta}_{a,b} = \frac{4}{\pi^2\eta^2 R(\eta)}\exp\bigg(-\frac{2}{R(\eta)}\{S(\eta)(|\alpha_a|^2 + |\beta_b|^2) - \sinh 2r(\alpha_a\beta_b + \alpha^*_a\beta^*_b)\}\bigg), \quad (11)$$

and its marginal single-mode Winger function is

$$W^{\eta}_{a} = \frac{2}{\pi\eta S(\eta)}\exp\bigg(-\frac{2|\alpha_a|^2}{S(\eta)}\bigg), \quad (12)$$

where $R(\eta) = 2(1 - 1/\eta)(1 - \cosh 2r) + 1/\eta^2$ and $S(\eta) = \cosh 2r - 1 + 1/\eta$. The expectation values of the EW operator (5) for two-mode squeezed states are shown in Fig. 1(b) with different squeezing rates r. It shows that our scheme allows one to detect some continuous variable entanglement with detector efficiency of about 40%. As shown in Fig. 1(c), violations of the inequality show different tendencies depending on efficiency η with increasing the squeezing parameter r. In the case of low squeezing rates the violation is maximized when $\eta = 0.5$, while for larger squeezing rates about $r \geqslant 1.2$ the violation is maximized when $\eta = 1$. This is because the dominant degree of freedom of entanglement detected by the observable in Eq. (1) changes decreasing efficiency η. Note that in the case $\eta = 0.5$ the dominant contribution to the entanglement arises from quantum correlations between the vacuum and the photon being present, while for $\eta = 1$ it comes from higher order correlations of photon number states.

VI. TESTING WITH *A PRIORI* ESTIMATED EFFICIENCY

So far it has been assumed that the detector efficiency is known precisely prior to the tests both in our scheme presented earlier in this article and in previous proposals [10,13,14]. This can be realized, for example, by a full characterization of detectors when doing a quantum tomography on the detectors which has been experimentally achieved [22]. However, in most cases *a priori* estimates of the detector efficiency ($\equiv\varepsilon$) may not be perfect and thus can be different from the real efficiency η that affects measured data. Let us assume that we can discriminate perfectly only whether the real efficiency $\eta > 1/2$ or $\eta \leqslant 1/2$. If $\eta \leqslant 1/2$, we can see that the EW in Eq. (8) is formulated only by experimentally measured Wigner functions. Thus, in this case our EW can be tested without knowing the real efficiency. On the other hand, for the case $\eta > 1/2$, the efficiency variable η is explicitly included in the EW (8) and should be replaced with the estimated efficiency ε as

$$|\langle\hat{\mathcal{W}}\rangle_{\eta>\frac{1}{2}}| = \bigg|\frac{\pi^2}{4\varepsilon^2}\big[W^{\eta}_{1,1} + W^{\eta}_{1,2} + W^{\eta}_{2,1} - W^{\eta}_{2,2}\big] + \frac{\pi(\varepsilon - 1)}{\varepsilon^2} \times \big[W^{\eta}_{a=1} + W^{\eta}_{b=1}\big] + 2\bigg(1 - \frac{1}{\varepsilon}\bigg)^2\bigg| \leqslant 2. \quad (13)$$

Note that Eq. (13) is valid subject to the condition

$$\eta(\text{real efficiency}) \leqslant \varepsilon \text{ (estimated efficiency)},$$

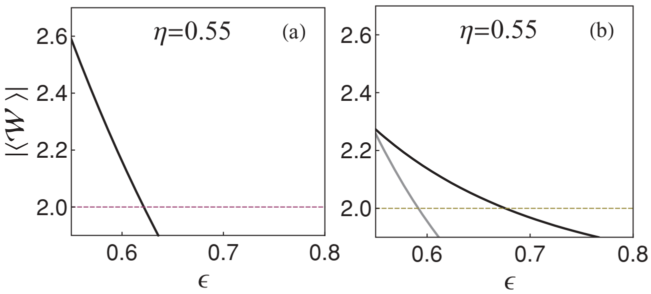

FIG. 2. (Color online) Witnessing entanglement with a real efficiency $\eta = 0.55$ when varying the estimated efficiency ε for an input of (a) a single-photon entangled state and (b) a two-mode squeezed state with $r = 0.4$ (black) and $r = 0.8$ (gray).

since otherwise the right-hand side of inequality in Eq. (13) is not valid, that is, the expectation values of separable states are not bounded by $\mathcal{W}^{\rm sep}_{\rm max} = 2$. In Fig. 2 we plot maximal values of the left-hand side of Eq. (13) against the estimated efficiency ε for single-photon entangled and two-mode squeezed states. The figure shows that entanglement can be detected even if this efficiency is estimated imperfectly. For example, an estimated efficiency of $\varepsilon = 0.65$ for a detector with real efficiency $\eta = 0.55$ allows the detection of entanglement of a two-mode squeezed state with $r = 0.4$ as shown in Fig. 2(b). The value of the left-hand side of Eq. (13) decreases as the gap between the real and estimated efficiencies increases. Nevertheless, any violation of the inequality (13) guarantees the existence of entanglement irrespective of the accuracy of the estimate.

VII. CONCLUSIONS

We have formulated an EW that allows one to detect entanglement even with significantly imperfect detectors. The proposed EW in (8) can be used to test arbitrary quantum states represented in phase space using the Wigner function. It can be implemented by both homodyne [8,17] and number-counting tomography methods [9] without additional steps for correcting measurement errors. Moreover, since the required minimal efficiency of our scheme is as low as 40%, it may be realizable using current detection technologies. In addition, our EW can be used without knowing the detection efficiency precisely prior to the test, which may allow more realistic implementation. We note that our approach is applicable to, for example, cavity QED or ion trap systems with the help of the direct measurement scheme for Wigner functions in such systems [23]. It will also be valuable to apply our scheme to quantum cryptography in which witnessing entanglement is a primal step for secure quantum key distribution [24]. We expect that our scheme enhances the possibility of witnessing entanglement in complex physical systems using current photo-detection technologies.

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