Adaptive Compressive Tomography with No *a priori* Information

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Quantum state tomography is both a crucial component in the field of quantum information and computation and a formidable task that requires an incogitable number of measurement configurations as the system dimension grows. We propose and experimentally carry out an intuitive adaptive compressive tomography scheme, inspired by the traditional compressed-sensing protocol in signal recovery, that tremendously reduces the number of configurations needed to uniquely reconstruct any given quantum state without any additional *a priori* assumption whatsoever (such as rank information, purity, etc.) about the state, apart from its dimension.

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Introduction.—The characterization of an unknown (true) quantum state $\rho_t \geq 0$ of Hilbert-space dimension d is a subject of immense study in quantum information [1–3]. To fully reconstruct an *arbitrary* ρ_t, one may perform a set of measurements that is enough to characterize all $d^2 - 1$ independent parameters that define ρ_t. Unfortunately, the number of such measurements generally grows polynomially with d, or exponentially with the number of subsystems that determine the quantum-source complexity. This poses a technical limitation on how far conventional quantum tomography can go in practical experiments [4,5].

If we know *a priori* that rank $\{\rho_t = \rho_r\} \leq r$ is extremely small, $r \ll d$, then the concept of compressed sensing (CS), whose foundation was first mathematically laid in the context of imaging [6–8], facilitates the search for a unique estimator by measuring much fewer configurations [9–12]. We say that the corresponding data are *informationally complete (IC)* for ρ_r. The state-of-the-art CS measurements to be performed given such prior information have been constructed in Ref. [13].

The standard CS procedure, nevertheless, has two important issues that need to be addressed. First, an *a priori* knowledge about r is necessary to establish a preliminary order-of-magnitude estimate for the number of configurations needed to fully characterize ρ_r of rank no larger than r. Accuracy of the final estimator is hence highly dependent on the validity of this *a priori* guess. Second, one has no means of verifying whether the measurement data at hand are truly IC for ρ_r in the standard scheme. Typically, accuracy surveys with target states are employed [10–12] and the value of such a survey relies on the precision of these target states. Therefore, the decision of *a priori* rank information and presumed choices of target states are ultimately debatable in the presence of experimental errors, rendering the reliability of any related tomography scheme questionable.

In this Letter, we establish a new adaptive tomography paradigm that completely removes the need for any sort of *a priori* information about ρ_r (except for its dimension d). Our proposed *adaptive compressive tomography* (ACT) also includes an efficient recipe to determine informational completeness of the collected data. No target states are ever required to validate the resulting state estimator. The convex boundary of the quantum state space and the positivity constraint plays the principal role in checking whether the accumulated data are IC and adaptively choosing measurements efficiently to uniquely reconstruct ρ_r, the two of which completely define the purpose of ACT.

To demonstrate ACT, we perform an experiment with the orbital angular momentum (OAM) of single photons and apply ACT to states of various ranks engineered in these degrees of freedom. Both experimental and simulated results show that ACT requires a smaller number of measurements to reconstruct rank-deficient quantum states as compared to conventional CS tomography with known types of CS measurements.

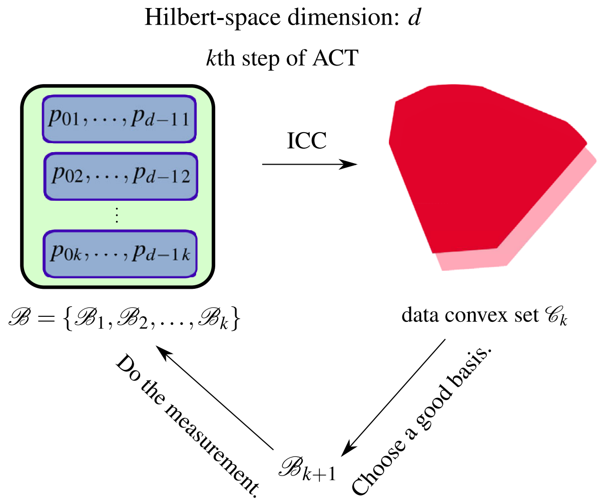

FIG. 1. Schematic diagram of a particular adaptive step in ACT tomography. In a clockwise flow, ACT first performs ICC to check whether data (blue) collected from measuring $\mathcal{B}$ are IC or not. If not, it proceeds to choose a good basis to measure in the next step.

The quantum state space and ACT.—In the absence of statistical fluctuations, we measure a randomly chosen computational basis $\{|0\rangle,|1\rangle,\ldots,|d-1\rangle\}$ on ρ_r of Hilbert-space dimension d. The corresponding Born probabilities $p_j=\langle j|\rho_r|j\rangle$ $(0\leq j\leq d-1)$ specify only the diagonal elements of ρ_r, and there is in principle a *data convex set* $\mathcal{C}=\{\rho|\rho\leftrightarrow p_j\ \ \forall\ \ j\}$ comprising infinitely many estimators $\hat{\rho}$ that are consistent with p_j. Evidently, $\rho_r\in\mathcal{C}$, and so the only fundamental objective of ACT is to shrink $\mathcal{C}$ to a single point with only $k_{\mathrm{IC}}\ll d+1$ IC measurement bases for $r\ll d$. For noiseless situations, this point must be ρ_r.

To gain insights into how quantum positivity constraint plays a major role in shrinking $\mathcal{C}$, we argue in the Supplemental Material (Sec. I [14]) that if one methodically measures $k_0=\lceil(r^2-r)/(d-1)\rceil+1$ orthonormal bases, one of which being the eigenbasis $\mathcal{B}_{\rho_r}$ of ρ_r, then $\hat{\rho}=\rho_r$ is the *unique positive estimator* consistent with all measured probabilities. For $r\ll d$, the regime of our interest, $k_0=2$. It is however clear that $k_{\mathrm{IC}}>2$ in real-world settings where ρ_r is completely unknown (apart from its dimension d), so the famous no-go answer to Pauli's phase-retrieval problem [15,16] still stands. Regardless, the positivity constraint can still ensure an efficient compression of the IC tomography procedure solely by data analysis.

The goal of ACT is to uniquely reconstruct any given unknown ρ_r through adaptively measuring one orthonormal basis at a time according to collected data, as sketched in Fig. 1. In the kth step of the adaptive scheme, ACT performs two main procedures. (1) First, ACT checks whether the probabilities $p_{j'k'}=\mathrm{tr}\{\rho_r\Pi_{j'k'}\}$ obtained from outcomes $\Pi_{j'k'}>0$ $(\Pi_{j'k'}\Pi_{j''k'}=\delta_{j',j''})$ of all the measured orthonormal bases $\mathcal{B}=\{\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_k\}=\{\Pi_{01},\ldots,\Pi_{d-11},\ldots,\Pi_{0k},\ldots,\Pi_{d-1k}\}$ so far are IC. Since the *accumulated* data define a data convex set $\mathcal{C}_k$ of size s_k that contains all quantum states ρ consistent with $p_{j'k'}$, this procedure is tantamount to finding out whether s_k is zero or not. If $s_{k=k_{\mathrm{IC}}}=0$, then the estimator $\hat{\rho}_{k=k_{\mathrm{IC}}}\geq 0$ consistent with the IC data is unique by definition, and equal to ρ_r when statistical fluctuation is absent. (2) If $s_k\neq 0$, the accumulated data collected are not IC and ACT shall choose the next basis by analyzing $\mathcal{C}_k$. Beginning with $k=1$, a "good" adaptive bases sequence should lead to a quick convergence of $\mathcal{C}\to\rho_r$ as ACT progresses.

Informational completeness certification (ICC).—To certify whether all collected data are IC or not in the kth adaptive step, it suffices to note that since $\mathcal{C}_k$ is convex, maximizing and minimizing the linear function $f_Z(\rho)=\mathrm{tr}\{\rho Z\}$ for some operator Z over $\rho\in\mathcal{C}_k$, respectively, give unique solutions $\rho_{\max}$ and $\rho_{\min}$ to the corresponding maximum $f_{\max,k}$ and minimum $f_{\min,k}$. Without loss of generality, Z is taken to be a random full-rank state. We may define the quantity $s_{\mathrm{CVX},k}=(f_{\max,k}-f_{\min,k})/(f_{\max,1}-f_{\min,1})$ that is a *size monotone* (see Sec. II of Supplemental Material [14]) for $\mathcal{C}_k$ in the sense that $s_k<s_{k-1}$ if $s_{\mathrm{CVX},k}<s_{\mathrm{CVX},k-1}$—it is a witness for the shrinkage of $\mathcal{C}_k$. As more linearly independent bases are measured, $s_{\mathrm{CVX},k}\geq s_{\mathrm{CVX},k+1}$ and $s_{\mathrm{CVX},k_{\mathrm{IC}}}=0$ implies that $s_{k_{\mathrm{IC}}}=0$ and that all data collected are IC for a unique reconstruction of ρ_r. Therefore, at every adaptive step in ACT, we run: ICC in the kth step. (1) Maximize and minimize $f_Z(\rho)=\mathrm{tr}\{\rho Z\}$ for a fixed, randomly chosen full-rank state $Z\neq 1/d$ to obtain $f_{\max,k}$ and $f_{\min,k}$ subject to the following: (i) $\rho\geq 0$, $\mathrm{tr}\{\rho\}=1$, (ii) $\mathrm{tr}\{\rho\Pi_{j'k'}\}=p_{j'k'}^{(\mathrm{ML})}$ for $0\leq j'\leq d-1$ and $1\leq k'\leq k$. (2) Compute $0\leq s_{\mathrm{CVX},k}\leq 1$ and check if it is smaller than some threshold ϵ. (3) If $s_{\mathrm{CVX},k}<\epsilon$, terminate ACT. Continue otherwise.

The aforementioned strategy is, as a matter of fact, a semidefinite program (SDP) [17] that can be efficiently solved by a variety of numerical methods. We should clarify here that while determining whether a set of measurement bases $\mathcal{B}$ possesses the conventional CS property for the entire class of rank-r states is an NP-hard problem [18], ascertaining whether $\mathcal{B}$ gives a unique estimator for one unknown ρ_r with the measurement data is, on the other hand, only as computationally difficult as carrying out the semidefinite program in ICC with a worst-case polynomial complexity [17].

For experimental data $\sum_{j'}\nu_{j'k'}=1$ $(1\leq k'\leq k)$ with statistical noise, $\mathcal{C}_k$ is defined as the *maximum-likelihood (ML) convex set* in which all $\rho\in\mathcal{C}_k$ satisfy the physical constraints $p_{j'k'}^{(\mathrm{ML})}=\mathrm{tr}\{\rho\Pi_{j'k'}\}$ imposed by the ML principle for quantum states $[p_{j'k'}^{(\mathrm{ML})}\to p_{j'k'}$ for $N\to\infty]$ [2,3,19,20]. All arguments for noiseless data hold exactly for the ML probabilities, so that the working principle of ICC is *perfectly robust against arbitrary noise* in the sense that $s_{\mathrm{CVX},k_{\mathrm{IC}}}=0$ still implies $s_{k_{\mathrm{IC}}}=0$ for noisy data owing to the preserved convexity of the newly defined $\mathcal{C}_k$. Noise only affects the reconstruction accuracy of the final unique estimator relative to ρ_r, which is a different subject matter for discussion.

Adaptive selection of measurement bases.—The optimal orthonormal basis to pick in the kth step and measure in the $(k+1)$th step is the one that minimizes $s_{\mathrm{CVX},k+1}$. Since ρ_r is unknown, we can treat some *a posteriori* estimator $\hat{\rho}_k$ from $\mathcal{C}_k$ as a guess for ρ_r to generate simulated data during the minimization of $s_{\mathrm{CVX},k+1}$ over all future basis choices. The complicated dependence of $s_{\mathrm{CVX},k+1}$ on the future basis however makes its brute-force optimization computationally exhaustive for large d.

For a more tractable approach to adaptively measure good bases, we first note that $\mathcal{C}_{k<k_{\mathrm{IC}}}$ *essentially* contains states with eigenbases that are distinct from $\{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$ (see Sec. III of the Supplemental Material [14]). So even if we know nothing about ρ_r, if it is rank-deficient, then taking $\mathcal{B}_{k+1}$ to be the diagonal basis of a rank-deficient $\hat{\rho}_k \in \mathcal{C}_k$ ensures a distinct measurement basis in each step that generates a reasonably fast converging sequence $\mathcal{B}_k \to \mathcal{B}_{\rho_r}$ as k increases since $\mathcal{C}_k \to \rho_r$ at the same time. There is more than one approach to pick eigenbases of rank-deficient states from $\mathcal{C}_k$, and as an example we shall consider the minimization of von Neumann entropy function $S(\rho) = -\mathrm{tr}\{\rho \log \rho\}$. A superfast algorithm suitable for minimizing S over $\mathcal{C}_k$ exists [20,21]. Incidentally, it was reported in Refs. [22,23] that entropy minimization also offers high compressive efficiencies in both sparse-signal and low-rank matrix recovery.

Complete ACT protocol.—All aforementioned arguments can accommodate real experimental situations, where the relative frequency data do not typically correspond to physical quantum states for $k > 1$. The data convex sets contain states that are now consistent with the corresponding physical ML probabilities derived from data, which are statistically consistent with the true probabilities. The final unique estimator $\hat{\rho}_{k_{\mathrm{IC}}}$ would then incur a statistical bias from ρ_r that drops as N increases. For many-body quantum sources, the bases generated by ACT are entangled. In practice, product bases are typically much more practical to implement for such sources. While verifying if a rank-deficient $\hat{\rho}_k \in \mathcal{C}_k$ can possess a product eigenbasis is computationally difficult, ACT can still be adjusted to feasibly generate near-optimal product bases (PACT) by defining $\mathcal{B}_{k+1}$ to be the product basis that is nearest to the eigenbasis of $\hat{\rho}_k$ with respect to some given norm using a nonlinear optimization routine. Both ACT and PACT for any experimental setting are summarized as follows: Beginning with $k = 1$ and a random computational basis $\mathcal{B}_1$: (1) measure $\mathcal{B}_k$ and collect the relative frequency data $\sum_{j'=0}^{d-1} \nu_{j'k} = 1$. (2) From $\{\nu_{0k'}, \ldots, \nu_{d-1k'}\}_{k'=1}^{k}$, obtain kd physical ML probabilities. (3) Perform ICC with the ML probabilities and compute $s_{\mathrm{CVX},k}$: (i) *If* $s_{\mathrm{CVX},k} < \varepsilon$, terminate ACT and take $\rho_{\max} \approx \rho_{\min}$ as the estimator and report $s_{\mathrm{CVX},k}$. (ii) *Else* Proceed. (4) Choose a rank deficient $\hat{\rho}_k \in \mathcal{C}_k$ [for instance by minimizing the von Neumann entropy $S(\rho)$ in $\mathcal{C}_k$]. (5) Define $\mathcal{B}_{k+1}$ to be the eigenbasis of $\hat{\rho}_k$ for ACT, or a basis close to it for PACT *via* some prechosen distance minimization technique. (6) Set $k = k + 1$ and repeat.

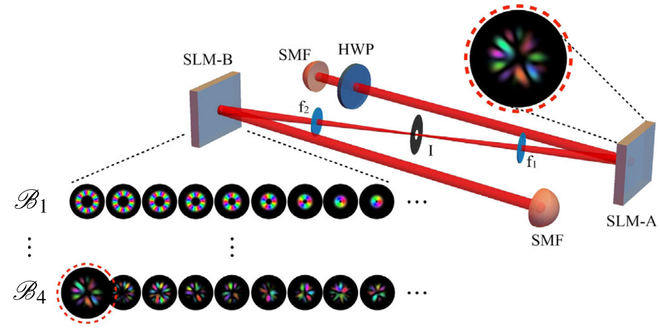

FIG. 2. Schematic of the OAM-based experimental setup. A 16-dimensional OAM state is generated at SLM-A using a holographic technique that allows the tailoring of the intensity and phase profile of the incoming beam. The modulated first-order of diffraction is filtered out using an iris (I) and a pair of lenses (f_1 and f_2). A similar holographic technique is used at the second SLM-B to measure the state in a given basis. The first measurement basis, $\mathcal{B}_1$, is given by the OAM computational basis. In the case of the rank 1 state shown on SLM-A, the corresponding eigenbasis is achieved after the fourth iteration.

Analysis and experiments.—We put both ACT and PACT tomography schemes to the experimental test by comparing their results with those from measuring random Pauli (RP) bases considered in Refs. [10–12], the Baldwin-Goyeneche (BG) bases in Ref. [13] that generalizes a known five-bases construction for $r = 1$ to an IC set of $k_{\mathrm{IC}} = 4r + 1$ bases for rank-r quantum states, and the set of random orthonormal bases of $k_{\mathrm{IC}} \approx \lceil 4r(d-r)/(d-1) \rceil$ studied in Ref. [24]. This exact scaling shall be used to benchmark the experimental k_{IC}s.

To demonstrate all three schemes (see Fig. 2), we experimentally emulate a four-qubit ($d = 16$) quantum system and both entangled and product measurement bases using an OAM-based setup. In particular, we consider the Laguerre-Gauss (LG) modes with azimuthal and radial mode indices ℓ and $p = 0$, respectively. Hence, OAM states correspond to a subspace of the LG modes and are characterized by a helical wave front given by $e^{i\ell\phi}$, where ℓ is the azimuthal index that corresponds to the OAM value, and ϕ is the azimuthal coordinate. The appropriate phase and intensity patterns are realized using a holographic technique called *intensity masking*, which is readily achieved by a programmable spatial light modulator (SLM) [25]. By doing so, we can prepare *any* many-body state and measurement basis. The generated photons are detected using the projective technique of *intensity flattening* [26], where any arbitrary spatial mode can be measured using an SLM followed by a single mode fiber (SMF).

A heralded single photon source is achieved by pumping a 3 mm β-barium borate type I nonlinear crystal with a quasicontinuous wave laser at a wavelength of 355 nm,

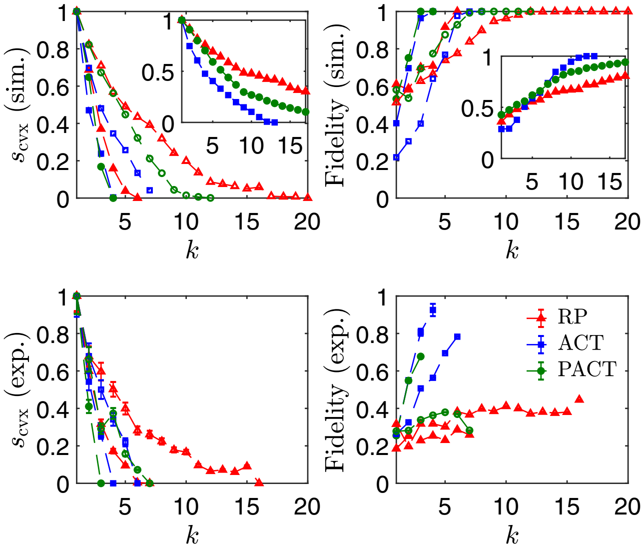

FIG. 3. Plots of simulation (noiseless) and experimental values of s_{CVX} and $\hat{\rho}_k$ fidelity against the measured basis number k for $d = 16$, where $\hat{\rho}_k \coloneqq \rho_{\min}$ for the RP scheme. Estimated experimental error bars reflect the propagated Poissonian-source standard deviations. All plot markers are averaged over five ρ_rs. The filled markers represent results for rank-1 ρ_rs whereas the unfilled ones represent those for rank-3 ρ_rs. The insets showcase simulation performances of rank-6 states as a demonstration of high-rank $(r \approx D/2)$ compressive tomography, with $1 \leq k \leq (D+1) = 17$ restricted to the minimal bases number for arbitrary-state tomography. The lower experimental fidelities for RP and PACT are due to a technical bias of the OAM setup for finite N, where bases close to the eigenbasis of ρ_r tend to give estimated Born probabilities that are relatively more accurate than those that are not. So for OAM sources, ACT is the most favorable option, as both PACT and RP correspond to measurement bases that are never close to the eigenbasis of ρ_r. Even with noisy data, ICC can still validate whether the resulting ML probabilities obtained from data are IC (left panels), which is the point of ACT.

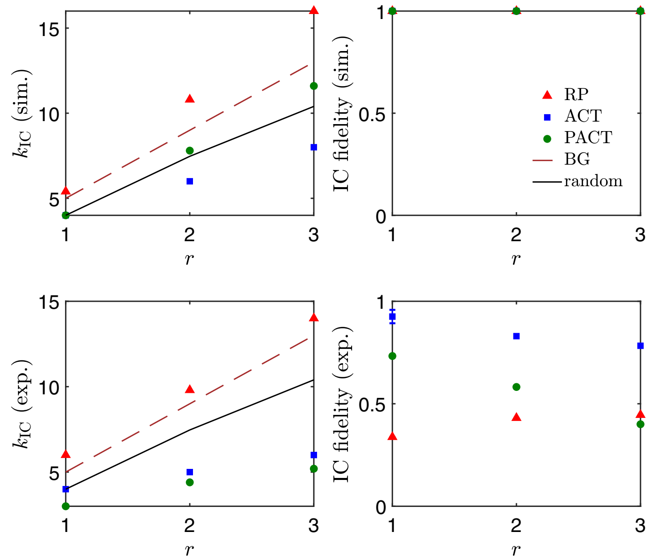

FIG. 4. Plots of simulation (noiseless) and experimental values of k_{IC} and the $\hat{\rho}_{k=k_{\mathrm{IC}}}$ fidelity against the rank $1 \leq r \leq 3$ of ρ_r. All k_{IC} values ascertained using ICC are averaged over five ρ_ts per rank. Otherwise all specifications are the same as Fig. 3. Although, for real data, positivity modifies the k_{IC} performances with ML, and PACT achieves informational completeness much quicker than RP as far as local bases are concerned. A comparison with random IC orthonormal bases shows that ACT gives a much lower value owing to the additional assessment of and optimization over $\mathcal{C}$.

producing photon pairs at 710 nm via spontaneous parametric down-conversion. A coincidence rate of 40 kHz, within a coincidence time window of 5 ns, is measured after filtering the photons to the fundamental Gaussian modes using SMF. Subsequent to the generation and detection of the photonic states, explained above, coincidence measurements are recorded using single photon detectors and a coincidence logic.

All results are summarized in Figs. 3 and 4, and the messages conveyed are succinctly stated here: for noiseless simulated data, in terms of *average* k_{IC} over uniformly (Hilbert-Schmidt) distributed rank-r true states, ACT is the most efficient, since it guides the measurement basis to the eigenbasis of ρ_r. The more many-body-suited PACT that adaptively generates product bases requires a larger k_{IC} to yield IC data, but the average performance margin with ACT is narrow for low-r states and is on par with the scaling of entangled Goyeneche-type bases ($k_{\mathrm{IC}} = 4r + 1$) for larger r. RP turns out to be least efficient amongst all tested schemes. Even in the presence of real data noise, both ACT and PACT remain the more favorable candidates for tomography on general complex systems.

Concluding remarks.—The feasible concept of adaptive compressive tomography developed here provides a powerful method to reconstruct any unknown rank-deficient quantum state with optimally chosen entangled or product orthonormal measurement bases, especially for quantum sources of complex degrees of freedom, which includes many-body systems. More importantly, the adaptive scheme requires no *a priori* knowledge or assumptions about the state or near-proximity target states because it can self-sufficiently validate whether the measured data are informationally complete or not using semidefinite programming, so that reliable compressive tomography can now be carried out in real experimental situations with noisy data. The superior compressive efficiencies of both entangled and product versions of our adaptive schemes are confirmed experimentally and demonstrated with respect to other established protocols.

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