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PAPER

Optimizing nontrivial quantum observables using coherence

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Abstract

In this paper we consider the quantum resources required to maximize the mean values of any nontrivial quantum observable. We show that the task of maximizing the mean value of an observable is equivalent to maximizing some form of coherence, up to the application of an incoherent operation. For any nontrivial observable, there always exists a set of preferred basis states where the superposition between such states is always useful for optimizing the mean value of a quantum observable. The usefulness of such states is expressed in terms of an infinitely large family of valid coherence measures which is then shown to be efficiently computable via a semidefinite program. We also show that these coherence measures respect a hierarchy that gives the robustness of coherence and the l_1 norm of coherence additional operational significance in terms of such optimization tasks.

1. Introduction

Quantum coherence has long been recognized as a fundamental aspect of quantum mechanics. In comparison, the identification of quantum coherence as a useful and quantifiable resource is a much more recent development. Progress in this area has been greatly accelerated due to the resource theoretical framework for quantum coherence [1–3]. Inspired by the resource theory of entanglement [4, 5], the notion of what quantum coherence is and how it can be quantified is now axiomatically defined, thus allowing quantum coherence phenomena to be discussed much more unambiguously. Since this development, many coherence measures have been proposed. Some known measures include geometric measures [2], the robustness of coherence [6–8], and entanglement based measures [9]. Coherence measures have now been studied in relation to a diverse range of quantum effects such as quantum interference [10], exponential speed-up in quantum algorithms [11, 12] and quantum metrology [13, 14], nonclassical light [15–17], quantum macroscopicity [18, 19] and quantum correlations [20–25]. An overview of coherence measures and their structure may be found in [26, 27]. Also related is the study of coherence witnesses, which concerns the detection, but not necessarily the quantification, of quantum coherences via an observable [6–8, 10, 28].

In this paper, we discuss how to construct a coherence measure from a quantum observable M. The structure of this paper follows: in section 2 we briefly review some essential concepts such as the Kraus and Choi–Jamiolkowski representations of quantum channels, the resource theory of coherence, and semidefinite programming. In section 3, we show that optimizing the mean value of the observable $\langle M\rangle$ for an input state is the same as maximizing the coherence of the input state pertaining to a specific class of bases, up to the application of some incoherent operation. Given any nontrivial observable M, it is therefore always possible to construct a coherence measure for some specific set of bases. We also show that the converse is possible, by identifying observables M and constructing a coherence measure for any given basis. In section 4, we prove that this measure is computable via a semidefinite program. In section 5, we demonstrate that the robustness of coherence and the l_1 norm of coherence establishes the quantum limits of such tasks. The relationship between our proposed measures and coherence witnesses is also discussed. In section 6, we present examples that illustrate the key ideas of our approach and provide several examples of previously known measures that turn

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out to be special cases of our proposed measures. Finally, in section 7 we summarize and discuss the implications of our results.

2. Preliminaries

We review some elementary concepts concerning coherence measures, quantum channels, and semidefinite programs.

We first briefly describe the formalism of quantum channels, which we take here to mean the set of all Completely Positive, Trace Preserving (CPTP) maps. There are several equivalent characterizations of quantum maps, but for our purposes, we will be concerned with the Kraus [29] and the Choi–Jamiolkowski representations [30, 31]. In the Kraus representation, a quantum operation is represented by a map of the form $\Phi(\rho) = \sum_i K_i \rho K_i^\dagger$ which is completely specified by a set of operators $\{K_i\}$ called Kraus Operators. The Kraus operators must satisfy the completeness relation $\sum_i K_i^\dagger K_i = \mathbb{1}$ in order to qualify as a valid quantum operation. In the Choi–Jamiolkowski representation, a quantum map Φ is represented by an operator $J(\Phi) = \sum_{i,j} \Phi(|i\rangle_A \langle j|) \otimes |i\rangle_B \langle j|$ which satisfies $\mathrm{Tr}_A[J(\Phi)] = \mathbb{1}_B$. The action of Φ on some state ρ is then recovered via the map $\mathrm{Tr}_B[J(\Phi)\mathbb{1}_A \otimes \rho_B^T] = \Phi(\rho_A)$. A simple relationship connects both equivalent representations. For a map Φ represented by Kraus operators $\{K_i = \sum_{j,k} k_{i,j,k} |j\rangle \langle k|\}$, the corresponding Choi–Jamiolkowski representation is $J(\Phi) = \sum_i v_i v_i^\dagger$ where $v_i := \sum_{j,k} k_{i,j,k} |j\rangle |k\rangle$.

The notion of coherence that we will employ in this paper will be the one identified in [1, 2], where a set of axioms are identified in order to specify a reasonable measure of quantum coherence. The axioms are as follows:

For a given fixed basis $\{|i\rangle\}$, the set of incoherent states $\mathcal{I}$ is the set of quantum states with diagonal density matrices with respect to this basis. Incoherent completely positive and trace preserving maps (ICPTP) are quantum maps that map every incoherent state to another incoherent state. Consider some set of ICPTP maps $\mathcal{O}$. Given this, we say that $\mathcal{C}$ is a measure of quantum coherence if it satisfies following properties: (C1) (Faithfulness) $\mathcal{C}(\rho) \geqslant 0$ for any quantum state ρ and equality holds if and only if $\rho \in \mathcal{I}$. (C2a) (Weak monotonicity) $\mathcal{C}$ is non-increasing under any ICPTP map $\Phi \in \mathcal{O}$, i.e. $C(\rho) \geqslant C(\Phi(\rho))$. (C2b) (Strong monotonicity) $\mathcal{C}$ is monotonic on average under selective outcomes, i.e. for any ICPTP map $\Phi \in \mathcal{O}$ such that $\Phi(\rho) = \sum_n K_n \rho K_n$, $C(\rho) \geqslant \sum_n p_n C(\rho_n)$, where $\rho_n = K_n \rho K_n^\dagger / p_n$ and $p_n = \mathrm{Tr}[K_n \rho K_n^\dagger]$ for all K_n with $\sum_n K_n K_n^\dagger = \mathbb{1}$ and $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$. (C3) (Convexity) $\mathcal{C}$ is convex, i.e. $\lambda C(\rho) + (1-\lambda) C(\sigma) \geqslant C(\lambda\rho + (1-\lambda)\sigma)$, for any density matrix ρ and σ with $0 \leqslant \lambda \leqslant 1$.

One may check that a particular operation is incoherent if its Kraus operators always maps a diagonal density matrix to another diagonal density matrix. One important example of such an operation is the CNOT gate. We can also additionally distinguish between the maximal set of ICPTP maps, which we refer to as maximally incoherent operations (MIO) [1] from the set of ICPTP maps whose Kraus operators additionally satisfy $K_n \mathcal{I} K_n^\dagger \subseteq \mathcal{I}$, which we refer to as simply incoherent operations (IO) [2]. From this definition, it is clear that IO $\subset$ MIO. We highlight that both MIO and IO are commonly used abbreviations, and that other possible sets of ICPTP maps are also actively being considered (see [26] for examples). In this article, we will typically consider either MIO and IO for the set $\mathcal{O}$.

Finally, we review some basic notions regarding semidefinite programs. A semidefinite program is a linear optimization problem over the set of positive matrices X, subject to a set of constraints that can be expressed in the following form:

$$\begin{aligned} &\max_{X \geqslant 0} && \mathrm{Tr}(AX) \\ &\text{subject to} && \phi_i(X) = B_i, \quad i = 1, \ldots, m, \end{aligned}$$

where A and B_i are Hermitian matrices and ϕ_i is a linear, Hermiticity preserving map (i.e. it maps every Hermitian matrix to another Hermitian matrix) representing the ith constraint. The above is called the primal problem. The optimal solution to the primal problem is always upper bounded by the optimal solution to the dual problem, when they exist. The dual problem may be written as the following optimization problem over all possible Hermitian matrices Y_i:

$$\begin{aligned} &\min_{\{Y_i = Y_i^\dagger\}} && \sum_{i=1}^m \mathrm{Tr}(B_i Y_i) \\ &\text{subject to} && \sum_{i=1}^m \phi_i^*(Y_i) \geqslant A. \end{aligned}$$

In this case, ϕ_i^* refers to the conjugate map that satisfies $\mathrm{Tr}[C^\dagger \phi_i(D)] = \mathrm{Tr}[\phi_i^*(C)^\dagger D]$ for every matrix C and D.

The solutions to the primal and dual problems are usually equal except in the most extreme cases. Nonetheless, this needs to be verified on a case by case basis. A sufficient condition for both primal and dual solution to be equal is called Slater's Theorem, which states that if the set of positive matrices X that satisfies all the constraints ϕ_i is nonempty, and if the set of Hermitian matrices Y_i that satisfies the strict inequality $\sum_{i=1}^{m} \phi_i^*(Y_i) > A$ is also nonempty, then the optimal solutions for both problems, also referred to as the optimal primal value and the optimal dual value, must be equal.

3. Coherence measures from maximally incoherent operations

In this section, we will discuss how a quantum observable M may be used to construct a coherence measure that satisfies axioms (C1)–(C3) (see section 2). The following theorem introduces a quantity that satisfies the strongly monotonic condition (C2b), which will prove useful when we eventually construct the coherence measure.

Theorem 1. *For any quantum observable M and quantum state ρ, the quantity*

$$\max_{\Phi \in \mathcal{O}} \mathrm{Tr}(M\Phi(\rho))$$

is strongly monotonic under incoherent operations, where $\mathcal{O}$ may be substituted with either the set of operations MIO or IO.

Proof. We first observe that any incoherent operation represented by some set of incoherent Kraus operators $\{K_i^{\mathrm{IO}}\}$ is, by definition, also a maximally incoherent operation. Note that for any set of maximally incoherent operations $\{\Omega_i^{\mathrm{MIO}} | \Omega_i^{\mathrm{MIO}} \in \mathrm{MIO}\}$, the map $\Omega(\rho) := \sum_i \Omega_i^{\mathrm{MIO}}(K_i^{\mathrm{IO}} \rho K_i^{\mathrm{IO}\dagger})$ is also maximally incoherent since it is just a concatenation of the incoherent operation represented by $\{K_i^{\mathrm{IO}}\}$, followed by performing a maximally incoherent operation Ω_i^{MIO} conditioned on the measurement outcome i. Let us assume that $\Omega_i^{\mathrm{MIO}}(\rho_i)$ is the optimal maximally incoherent operation maximizing $\mathrm{Tr}(M\Omega_i^{\mathrm{MIO}}(\rho_i))$ for the state $\rho_i := K_i^{\mathrm{IO}} \rho K_i^{\mathrm{IO}\dagger} / \mathrm{Tr}(K_i^{\mathrm{IO}} \rho K_i^{\mathrm{IO}\dagger})$, we then have the following series of inequalities:

$$\begin{aligned}
\max_{\Phi \in \mathrm{MIO}} \mathrm{Tr}(M\Phi(\rho)) &\geqslant \mathrm{Tr}(M\Omega(\rho)) \\
&= \mathrm{Tr}\,[M \sum_i \Omega_i^{\mathrm{MIO}}(K_i^{\mathrm{IO}} \rho K_i^{\mathrm{IO}\dagger})] \\
&= \mathrm{Tr}\,[M \sum_i p_i \Omega_i^{\mathrm{MIO}}(\rho_i)] \\
&= \sum_i p_i \max_{\Phi_i \in \mathrm{MIO}} \mathrm{Tr}(M\Phi_i(\rho_i)),
\end{aligned}$$

where $\rho_i := K_i^{\mathrm{IO}} \rho K_i^{\mathrm{IO}\dagger} / \mathrm{Tr}(K_i^{\mathrm{IO}} \rho K_i^{\mathrm{IO}\dagger})$ and $p_i := \mathrm{Tr}(K_i^{\mathrm{IO}} \rho K_i^{\mathrm{IO}\dagger})$. We note that the last line is simply the expression for strong monotonicity, which proves the result for the case when $\mathcal{O}$ is MIO. Identical arguments apply when considering IO, which completes the proof. □

In the above proof, we see that the optimization over MIO yields a valid coherence monotone in within the regime of IO, so drawing a sharp distinction between the two sets of operations is not always necessary.

We note that satisfying strong monotonicity qualifies the quantity as a coherence monotone, but is insufficient to fully qualify it as a coherence measure. In order for that to happen, we need to demonstrate that $\max_{\Phi \in \mathcal{O}} \mathrm{Tr}(M\Phi(\rho)) = 0$ iff ρ is an incoherent state, and $\max_{\Phi \in \mathcal{O}} \mathrm{Tr}(M\Phi(\rho)) > 0$ whenever ρ is a coherent state. It is clear that this is only true for some special cases of M. However, the following theorem shows that even if M does not by itself satisfy the above conditions, it is still possible to construct a valid coherence measure using M.

Theorem 2. *Let* M *be some Hermitian quantum observable in a* d*-dimensional Hilbert space. Then there exists a basis $\{|i\rangle\}$ such that $\langle i| \left(M - \frac{\mathrm{Tr}\,M}{d}\mathbb{1}\right)|i\rangle = 0$ for every $|i\rangle$.*

Furthermore, for every nontrivial quantum observable M*, the quantity*

$$\mathcal{C}_M^{\mathcal{O}}(\rho) := \max_{\Phi \in \mathcal{O}} \mathrm{Tr}[(M\Phi(\rho)] - \mathrm{Tr}(M) \big/ d$$

is always a valid coherence measure w.r.t. any basis $\{|i\rangle\}$ that satisfies $\langle i| \left(M - \frac{\mathrm{Tr}\,M}{d}\mathbb{1}\right)|i\rangle = 0$ for every $|i\rangle$. Since such a basis always exists, the coherence measure $\mathcal{C}_M^{\mathcal{O}}$ also always exists. The set of quantum maps $\mathcal{O}$ may be subtituted with either MIO or IO.

Proof. We begin by observing that the matrix $M' = M - \frac{\mathrm{Tr}\,M}{d}\mathbb{1}$ is trace zero. Since M' is nontrivial (not proportional to the identity operator), it implies that the sum of its positive eigenvalues and negative eigenvalues

must be exactly equal. Let $\vec{\lambda} = (\lambda_1,\ldots,\lambda_d)$ be the vector of eigenvalues of M' arranged in decreasing order. We recall the Schur–Horn theorem, which states that for every vector $\vec{v} = (v_1, \ldots, v_d)$, there exists a Hermitian matrix with the same vector of eigenvalues $\vec{\lambda}$, but with diagonal entries $\vec{v} = (v_1, \ldots, v_d)$ so long as the vectors satisfy the majorization condition $\vec{v} \prec \vec{\lambda}$. It is clear that the zero vector $\vec{v} = (0, \ldots, 0)$ always satisfies this condition. Therefore, there always exist a basis $\{|i\rangle\}$ for M' where the main diagonals are all zero, such that $\langle i|\ M'|i\rangle = 0$ for every $|i\rangle$, which proves the first part of the theorem. See proposition 1for an example of such a basis using mutually unbiased bases. Proposition 1 presents an alternative proof for the existence of such bases but it is important to note that not every basis that satisfies the condition $\langle i|\ M'|i\rangle = 0$ for every $|i\rangle$ is necessarily mutually unbiased.

Now, we proceed to prove that $\mathcal{C}_M^{\mathcal{O}}(\rho)$ is a coherence measure of with respect to the basis $\{|i\rangle\}$. The strong monotonicity condition is already satisfied due to theorem 1. The convexity of the measure is immediate from the linearity of the trace operation and the definition of $\mathcal{C}_M^{\mathcal{O}}$ as a maximization over MIO or IO. Therefore, we only need to establish the faithfulness property of the measure.

In order to prove this, recall that in the basis $\{|i\rangle\}$, the diagonal elements of M' is all zero. Therefore, there always exists some projection onto a 2 dimensional space M' such that the corresponding submatrix has the form $\begin{pmatrix} 0 & r \\ r^* & 0 \end{pmatrix}$. We can assume without loss of generality that the projection is onto the subspace $\{|0\rangle, |1\rangle\}$, since at this point, the numerical labelling of the basis is arbitary.

For some coherent quantum state ρ, there is at least one nonzero off-diagonal element. Since basis permutation is an incoherent operation, we can assume the nonzero off-diagonal element is ρ_{01}. In fact, we can assume that it is the only nonzero off-diagonal element as we can freely project onto the subspace spanned by $\{|0\rangle, |1\rangle\}$ and completely dephase the rest of the Hilbert space via an incoherent operation, which allows us to prove the general result by only considering the 2 dimensional case. Suppose this leads to a 2 dimensional submatrix of the form $\begin{pmatrix} p_1 & a \\ a^* & p_2 \end{pmatrix}$ where a is nonzero since ρ is coherent.

Directly computing $\mathrm{Tr}\begin{pmatrix} 0 & r \\ r^* & 0 \end{pmatrix}\begin{pmatrix} p_1 & a \\ a^* & p_2 \end{pmatrix}$, we get the expression $r^*a + a^*r = |ra|(\mathrm{e}^{\mathrm{i}\phi} + \mathrm{e}^{-\mathrm{i}\phi})$. This final quantity can always be made positive by performing the incoherent unitary that performs $|0\rangle \rightarrow |0\rangle$ and $|1\rangle \rightarrow \mathrm{e}^{-\mathrm{i}\phi}|1\rangle$ which is equivalent to making both a and r positive quantities. Since r is strictly positive as M' is a nontrivial matrix, this implies $ar > 0$ if ρ is a coherent state, so there always exists at least one incoherent operation Φ such that $\mathrm{Tr}[M'\Phi(\rho)] > 0$ for every coherent state ρ.

Finally, we just observe that M' has zero diagonal elements w.r.t. the basis $\{|i\rangle\}$, so $\mathrm{Tr}[M'\Phi(\rho)] = 0$ whenever ρ is incoherent and Φ is MIO or IO. This completes the proof. □

Theorem 2 above establishes several facts. First, observe that since $\mathcal{C}_M^{\mathcal{O}}(\rho)$ is a coherence measure and nonnegative, $\mathrm{Tr}[(M\rho)] - \mathrm{Tr}(M)/d$ can only be positive when ρ is coherent (the basis is specified by the theorem). This establishes that every nontrivial observable M is, in fact, a witness of *some* form of coherence. One just needs to subtract the constant $\mathrm{Tr}(M)/d$ from the mean value $\langle M\rangle$ to verify the presence of coherence.

Second, it establishes that if M is a coherence witness, then it can be interpreted as the lower bound of the *bona fide* coherence measure $\mathcal{C}_M^{\mathcal{O}}$. Recall that the measure $\mathcal{C}_M^{\mathcal{O}}$ quantifies the operational usefulness of a quantum state when one considers MIO or IO type quantum operations and the task is to maximize the mean value of a given observable M. Other examples of coherence measures with operational interpretations in terms of MIO or IO include the relative entropy of coherence, which quantifies the number of maximally coherent qubits you can distill using IO [32], as well as quantities considering how much entanglement and Fisher information can be extracted via MIO or IO [9, 14].

Third, theorem 2 defines the preferred incoherent bases where the coherence is useful for optimizing $\langle M\rangle$ and shows that such bases always exist. The following proposition states that the coherence with respect to any basis that is mutually unbiased with respect to the eigenbasis of the observable M will always satisfy the necessary condition in theorem 2.

Proposition 1. *Let $\{|\alpha_i\rangle\}$ be the complete set of eigenbases of some nontrivial quantum observable* M, *and let $\{|\beta_i\rangle\}$ be any complete basis that is mutually unbiased w.r.t. $\{|\alpha_i\rangle\}$. Then the basis $\{|\beta_i\rangle\}$ always satisfies $\langle\beta_i|\left(M - \frac{\mathrm{Tr}\,M}{d}\mathbb{1}\right)|\beta_i\rangle = 0$ for every $|\beta_i\rangle$.*

In other words, w.r.t. any mutually unbiased basis $|\beta_i\rangle$, the diagonal elements of $M - \frac{\mathrm{Tr}\,M}{d}\mathbb{1}$ is always zero.

Proof. Let the dimension of the Hilbert space be d. We then have $|\langle\beta_i|\ \alpha_j\rangle|^2 = \frac{1}{d}$. Since $\{|\alpha_i\rangle\}$ is the complete eigenbasis of M, $M = \sum_i \lambda_i|\alpha_i\rangle\langle\alpha_i|$ and $\langle\beta_i|\ M|\beta_i\rangle = \sum_j \frac{\lambda_j}{d} = \frac{\mathrm{Tr}\,M}{d}$. This implies that

$\langle\beta_i|(M - \frac{\mathrm{Tr}\,M}{d}\mathbb{1})|\beta_i\rangle = 0$ for every $i = 1, \ldots, d$, which is the required condition. Note that this can be considered an alternative proof of the first statement in theorem 2. □

In theorem 2, we established the existence of a coherence measure $\mathcal{C}_M^{\mathcal{O}}$ but the proof is not constructive in the sense that given the observable M, it does not immediately inform us of a procedure to obtain the basis $\{|i\rangle\}$ and corresponding measure $\mathcal{C}_M^{\mathcal{O}}$. Proposition 1 closes this gap. Given any observable M, one may obtain the eigenbasis, find another basis that is mutually unbiased with respect to this eigenbasis, and construct $\mathcal{C}_M^{\mathcal{O}}$. An overview of how to construct mutually unbiased bases can be found in [33]. Note that mutually unbiases bases are not the only kinds of bases satisfying theorem 2.

We now consider the reverse construction. Suppose instead of starting from a given observable M and inferring the basis for the coherence measure, we wish to begin with some basis $\{|i\rangle\}$ and construct an observable M with corresponding measure $\mathcal{C}_M^{\mathcal{O}}$ that quantifies the coherence in the basis $\{|i\rangle\}$. The method to do this also follows from theorem 2, as we can choose any Hermitian matrix to be M so long as the leading diagonals are zero. This is guaranteed to lead to a reasonable measure according to theorem 2. Such a matrix is easy to construct, as any arbitrary Hermitian matrix written in the basis $\{|i\rangle\}$ with its leading diagonal elements replaced with zero will suffice. This is summarized in the form of the following corollary.

Corollary 2.1. *Consider any complete basis $\{|i\rangle\}$ and any arbitrary Hermitian matrix* H *which has at least one nonzero off-diagonal element. Then using the matrix $M = H - \sum_i \langle i|\, H|i\rangle\, |i\rangle\langle i|$, the corresponding measure $\mathcal{C}_M^{\mathcal{O}}$ (see theorem 2) will always be a measure of the coherence w.r.t. the basis $\{|i\rangle\}$.*

4. A semidefinite program for computing coherence measures

Previously, we have considered both MIO and IO during the construction of our coherence measures. Here, we show that for MIOs, the corresponding coherence measure $\mathcal{C}_M^{\mathrm{MIO}}$ is efficiently computable via a semidefinite program. Many other quantities related to coherence may be phrased as a semidefinite program. For examples, see [6, 7, 34–37].

Let us first define the matrix $A := M_A \otimes \rho_B^T \otimes |1\rangle_C\langle 1|$ acting on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$. Furthermore, we will assume that $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = d$ and $\dim(\mathcal{H}_C) = 2$.

We now prove the following:

Theorem 3. *For any quantum observable* M*, the optimization problem*

$$\max_{\Phi\in\mathrm{MIO}} \mathrm{Tr}(M\Phi(\rho))$$

is equivalent to the semidefinite program

$$\begin{aligned}
\max_{X\geqslant 0} \quad & \mathrm{Tr}(AX) \\
\text{subject to} \quad & \mathrm{Tr}_{AC}(X\,|1\rangle_C\langle 1|) = \mathbb{1}_B \\
& \mathrm{Tr}_{BC}(X\mathbb{1}_A \otimes |i\rangle_B\langle i| \otimes |1\rangle_C\langle 1|) \\
& = \sum_{j=1}^{d} \mathrm{Tr}_{ABC}(X\,|j\rangle_A\langle j| \otimes |i\rangle_B\langle i| \otimes |2\rangle_C\langle 2|)|j\rangle_A\langle j| \\
& \forall\ i = 1, \ldots, d,
\end{aligned}$$

where $A := M_A \otimes \rho_B^T \otimes |1\rangle_C\langle 1|$.

Note that all the matrices here are assumed to be written in a basis of the type specified in theorem 2.

Proof. We begin by first noting that the matrix X can be written as the matrix

$$\begin{pmatrix} X_1 & * \\ * & X_2 \end{pmatrix}.$$

The $*$ indicates possible nonzero elements, but they do not appear in the objective function we are trying to optimize, nor do they appear within the linear constraints, so they can be arbitrary so long as $X \geqslant 0$. The matrix A written in matrix form looks like

$$\begin{pmatrix} M_A \otimes \rho_B^T & 0 \\ 0 & 0 \end{pmatrix}.$$

Computing $\mathrm{Tr}(AX)$, we get

$$\mathrm{Tr}(AX) = \mathrm{Tr}_A[\mathrm{Tr}_B(X_1 \mathbb{1}_A \otimes \rho_B^T) M_A].$$

Now, the constraint $\mathrm{Tr}_{AC}(X\,|1\rangle_C\langle 1|) = \mathbb{1}_B$ implies $\mathrm{Tr}_A(X_1) = \mathbb{1}_B$, so X_1 actually represents a valid quantum operation in the Choi–Jamiolkowski representation. This implies $\mathrm{Tr}(AX)$ has the form $\mathrm{Tr}_A[\Phi(\rho)M_A]$ for some valid quantum operation Φ.

All that remains is for us to prove that under the set of constraints

$$\begin{aligned}\mathrm{Tr}_{BC}(X\mathbb{1}_A \otimes |i\rangle_B\langle i| \otimes |1\rangle_C\langle 1|)\\ = \sum_j \mathrm{Tr}_{ABC}(X\,|j\rangle_A\langle j| \otimes |i\rangle_B\langle i| \otimes |2\rangle_C\langle 2|)\,|j\rangle_A\langle j|\end{aligned}$$

for all $i = 1, \ldots, d$ and $j = 1, \ldots, d$, Φ must be a maximally incoherent operation. We first note that the number $\mathrm{Tr}_{ABC}(X\,|j\rangle_A\langle j| \otimes |i\rangle_B\langle i| \otimes |2\rangle_C\langle 2|)$ is just the main diagonal elements of the matrix X_2, so it must be nonnegative since X is positive and X_2 is a principle submatrix of X. We can therefore rewrite the constraint as $\mathrm{Tr}_{BC}(X\mathbb{1}_A \otimes |i\rangle_B\langle i| \otimes |1\rangle_C\langle 1|) = \sum_j \lambda_{i,j}\,|j\rangle_A\langle j|$ where $\lambda_{i,j}$ is nonnegative. This necessarily means that every incoherent state $|i\rangle\langle i|$ is mapped to a diagonal state $\sum_j \lambda_{i,j}|j\rangle\langle j|$ under the quantum map represented by X_1, which defines maximally incoherent operations, and completes the proof. □

Given the primal problem in theorem 3, we can also write down the dual problem, which is detailed in the following corollary:

Corollary 3.1. *The dual to the primal problem in theorem 3 is the following optimization over all possible Hermitian* $\mathrm{Y_A}$ *and* $\mathrm{Y_B}$*:*

$$\begin{aligned}\min_{Y_B = Y_B^\dagger} \quad & \mathrm{Tr}(Y_B)\\ \text{subject to } & \mathbb{1}_A \otimes Y_B + \sum_{i=1}^{d} Y_A \otimes |i\rangle_B\langle i| \geqslant M_A \otimes \rho_B^T\\ & \langle j|\, Y_A\, |j\rangle_A \leqslant 0, \ \forall j = 1, \ldots, d.\end{aligned}$$

Furthermore, the optimal primal value is equal to the optimal dual value.

Proof. The first constraint in the primal problem can be written as $\phi(X) := \mathrm{Tr}_{AC}(X\,|1\rangle_C\langle 1|) = \mathbb{1}_B$. The conjugate map can be verified to be the map $\phi^*(Y_B) = \mathbb{1}_A \otimes Y_B \otimes |1\rangle_C\langle 1|$, since it satisfies $\mathrm{Tr}[Y_B\phi(X)] = \mathrm{Tr}[\phi^*(Y_B)X]$.

The rest of the constraints can be written as

$$\begin{aligned}\phi_i(X) := \mathrm{Tr}_{BC}(X\mathbb{1}_A \otimes |i\rangle_B\langle i| \otimes |1\rangle_C\langle 1|)\\ - \sum_j \mathrm{Tr}_{ABC}(X\,|j\rangle_A\langle j| \otimes |i\rangle_B\langle i| \otimes |2\rangle_C\langle 2|)\,|j\rangle_A\langle j| = 0.\end{aligned}$$

In this case the conjugate map is

$$\begin{aligned}\phi_i^*(Y_A^i) := Y_A^i \otimes |i\rangle_B\langle i| \otimes |1\rangle_C\langle 1|\\ - \sum_j \langle j|\, Y_A^i\, |j\rangle_A |j\rangle_A\langle j| \otimes |i\rangle_B\langle i| \otimes |2\rangle_C\langle 2|.\end{aligned}$$

Summing over the variable i, we have

$$\begin{aligned}\sum_i \phi_i^*(Y_A^i) := \sum_i Y_A^i \otimes |i\rangle_B\langle i| \otimes |1\rangle_C\langle 1|\\ - \sum_{i,j} \langle j|\, Y_A^i\, |j\rangle_A |j\rangle_A\langle j| \otimes |i\rangle_B\langle i| \otimes |2\rangle_C\langle 2|.\end{aligned}$$

.

The dual program can therefore be written as:

$$\begin{aligned}\min_{Y_B=Y_B^\dagger} \quad & \mathrm{Tr}(Y_B)\\ \text{subject to } & \mathbb{1}_A\otimes Y_B\otimes|1\rangle_C\langle 1| + Y_A\otimes\mathbb{1}_B\otimes|1\rangle_C\langle 1|\\ & -\sum_i Y_A^i\otimes|i\rangle_B\langle i|\otimes|1\rangle_C\langle 1|\\ & -\sum_{i,j}\langle j|\,Y_A^i\,|j\rangle_A|j\rangle_A\langle j|\otimes|i\rangle_B\langle i|\otimes|2\rangle_C\langle 2|\\ & \geqslant M_A\otimes\rho_B^T\otimes|1\rangle_C\langle 1|.\end{aligned}$$

The third line of the constraint is actually just $-\sum_{i,j}\langle j|\,Y_A^i\,|j\rangle_A|j\rangle_A\langle j|\otimes|i\rangle_B\langle i|\otimes|2\rangle_C\langle 2|\geqslant 0$, which is equivalent to the contraint that the main diagonal of Y_A^i is all negative. As such, the program can be further simplified to the following:

$$\begin{aligned}\min_{Y_B=Y_B^\dagger} \quad & \mathrm{Tr}(Y_B)\\ \text{subject to } & \mathbb{1}_A\otimes Y_B+\sum_i Y_A^i\otimes|i\rangle_B\langle i|\geqslant M_A\otimes\rho_B^T\\ & \langle j|\,Y_A^i\,|j\rangle_A\leqslant 0,\ \ \forall j=1,\ldots,d\end{aligned}$$

which is the form that was presented in the corollary. Finally, we just need to check that the primal and dual programs satisfies Slater's conditions. For the primal problem, the optimization is over all MIO's, so the primal feasible set is nonempty (for instance, we can just consider the Choi–Jamiolkowski representation of the identity operation, which also falls under MIO). Furthermore, there exists at least one set of Y_A^i and Y_B s.t. $\mathbb{1}_A\otimes Y_B+\sum_i Y_A^i\otimes|i\rangle_B\langle i|>M_A\otimes\rho_B^T$ since one can always set $Y_A^i=0$, and $Y_B=x\mathbb{1}_B$ where $x>\lambda_{\max}(M_A\otimes\rho_B^T)$ and $\lambda_{\max}(A)$ represents the largest eigenvalue of A. As such, Slater's conditions are satisfied and the primal optimal value is equal to the dual optimal value. □

5. Relation to robustness and l_1 norm of coherence

It was observed in [6, 7] that the robustness of coherence $\mathcal{C}_{\mathcal{R}}$, which may be interpreted as the minimal amount of quantum noise that can be added to a system before it becomes incoherent, is a coherence measure that is also simultaneously the mean value of an observable. That is, for any state ρ, there always exists some optimal witness W_ρ such that $\mathrm{Tr}(W_\rho\rho)=\mathcal{C}_{\mathcal{R}}(\rho)$. It was also demonstrated that the l_1 norm upper bounds the robustness, so $\mathcal{C}_{\mathcal{R}}(\rho)\leqslant\mathcal{C}_{l_1}(\rho)$. The coherence weight is another similar example where the measure is given by the mean value of an observable [37]. Note that for both the robustness of coherence and the coherence weight, the optimal observables depend on the state. The following theorem shows that both the robustness and the l_1 norms of coherence are fundamental upper bounds of $\mathcal{C}_M^{\mathcal{O}}$. In [28], it was also observed that when M is a witness that achieves its maximum value for the maximally coherent state, then $\mathcal{C}_M^{\mathrm{IO}}$ is upper bounded by the l_1 norm of coherence under certain normalization conditions.

Theorem 4 Hierarchy of coherence measures. *For any given state ρ and observable* M*, the following hierarchy of the coherence measures holds:*

$$\mathcal{C}_M^{\mathrm{IO}}(\rho)\leqslant\mathcal{C}_M^{\mathrm{MIO}}(\rho)\leqslant\mathcal{N}_M\mathcal{C}_{\mathcal{R}}(\rho)\leqslant\mathcal{N}_M\mathcal{C}_{l_1}(\rho),$$

where $\mathcal{N}_M:=\left|\lambda_{\min}(M)-\frac{\mathrm{Tr}\,M}{d}\right|$ and $\lambda_{\min}(M)$ is the smallest eigenvalue of the observable M*. Furthermore, all the inequalities are tight.*

Proof. In [6, 7], it was shown that $\mathcal{C}_{\mathcal{R}}(\rho)$ is equivalent to maximizing $\mathrm{Tr}\,\rho W$ over all Hermitian observables W, subject to the constraint that $W\geqslant-\mathbb{1}$ and that the diagonal entries of W are nonpositive. Note that our convention differs from the one presented in [6, 7] by a negative sign.

We always displace M and consider the matrix $M'=M-\frac{\mathrm{Tr}\,M}{d}\mathbb{1}$, and it is clear that a positive scaling factor does not fundamentally change $\mathcal{C}_M^{\mathcal{O}}$, i.e. $\mathcal{C}_{kM}^{\mathcal{O}}(\rho)=k\mathcal{C}_M^{\mathcal{O}}(\rho)$ for $k\geqslant 0$ where $\mathcal{O}$ is MIO or IO. As such, without any loss in generality, we can assume that M is a traceless matrix where the leading matrix elements are zero, and that its smallest eigenvalue is normalized such that $\lambda_{\min}(M)=-1$. This implies that $\mathcal{N}_M=1$. Observe that under these assumptions, M automatically satisfies the constraints on W that was described in the preceding paragraph.

Recall that $\mathcal{C}_M^{\mathcal{O}}(\rho):=\max_{\Phi\in\mathrm{MIO}}\mathrm{Tr}(M\Phi(\rho))$. Consider the quantity $\mathrm{Tr}(M\Phi(\rho))$ and let Φ^* be the conjugate map such that $\mathrm{Tr}(M\Phi(\rho))=\mathrm{Tr}(\Phi^*(M)\rho)$. Then, since Φ is a CPTP map and the conjugate map preserves the trace, we have $\lambda_{\min}(\Phi^*(M))=\min_\rho\mathrm{Tr}\,(\Phi(\rho)M)\geqslant\min_\rho\mathrm{Tr}\,(\rho M)=\lambda_{\min}(M)$ where the minimum is taken

over all possible density matrices. Furthermore, we see that as Φ is MIO or IO, the leading diagonals of $\Phi^*(M)$ must be zero if the leading diagonals of M are zero. This again comes from the definition of the conjugate map $\mathrm{Tr}(M\Phi(\rho)) = \mathrm{Tr}(\Phi^*(M)\rho)$. From this, we can determine that $\Phi^*(M)$ always satisfies the necessary constraints for W described above, and this is true for any Φ that is an ICPTP map, so we must have $\mathcal{C}_M^{\mathcal{O}}(\rho) \leqslant \mathcal{N}_M\mathcal{C}_{\mathcal{R}}(\rho)$.

It was already known that $\mathcal{C}_{\mathcal{R}}(\rho) \leqslant \mathcal{C}_{l_1}(\rho)$, and we must have that $\mathcal{C}_M^{\mathrm{IO}}(\rho) \leqslant \mathcal{C}_M^{\mathrm{MIO}}(\rho)$ since IO $\subset$ MIO, which leads to the final chain of inequalities

$$\mathcal{C}_M^{\mathrm{IO}}(\rho) \leqslant \mathcal{C}_M^{\mathrm{MIO}}(\rho) \leqslant \mathcal{N}_M\mathcal{C}_{\mathcal{R}}(\rho) \leqslant \mathcal{N}_M\mathcal{C}_{l_1}(\rho).$$

To see that the inequalities are tight, we need to demonstrate that there are cases of M and ρ where equality is achieved. It is already known that when the state ρ is the maximally coherent state $\mathcal{C}_{\mathcal{R}}(\rho) = \mathcal{C}_{l_1}(\rho)$ [6, 7]. Consequently, we also know that when ρ is maximally coherent $W_\rho = \sum_{i\neq j}|i\rangle\langle j|$ and we can choose $M = W_\rho$, which is enough to achieve $\mathcal{C}_M^{\mathrm{MIO}}(\rho) = \mathcal{N}_M\mathcal{C}_{\mathcal{R}}(\rho)$. Finally, we can also verify that $\mathcal{C}_M^{\mathrm{IO}}(\rho) = \mathcal{C}_M^{\mathrm{MIO}}(\rho)$ is achieved when the input state ρ is maximally coherent and we choose $M = \sum_{i\neq j}|i\rangle\langle j|$. Therefore, all the inequalities are tight. □

Here, we also briefly discuss the computational resources necessary to evaluate $\mathcal{C}_M^{\mathrm{MIO}}$ and $\mathcal{C}_R$, which are both computable via semidefinite programs. We note that the matrix sizes required to calculate $\mathcal{C}_R$ are $d \times d$ where d is the dimension of the Hilbert space. In comparison, the matrix sizes required for $\mathcal{C}_M^{\mathrm{MIO}}$ are $d^2 \times d^2$ due to the optimization over maximally incoherent operations via the Choi–Jamiolkowski representation. The problem sizes for $\mathcal{C}_M^{\mathrm{MIO}}$ and $\mathcal{C}_R$ are therefore scales with $O(d^4)$ and $O(d^2)$ respectively. As such, $\mathcal{C}_M^{\mathrm{MIO}}$ will generally take longer to compute than $\mathcal{C}_R$. However, both measures are computable in polynomial time as semidefinite programs can be computed to any finite accuracy in polynomial time [38].

6. Examples

In this section, we discuss examples that illustrate our key results. We first consider the simplest system consisting of a single spin. Let $\{|\uparrow\rangle, |\downarrow\rangle\}$ be the basis vectors corresponding to a spin pointing in the $+z$ and the $-z$ directions respectively. Suppose our measurement observable is a Pauli measurement along the x axis. In this case, our observable M is the Pauli matrix $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Notice that σ_x always is always zero along its leading diagonals, as was the case considered in theorem 2. Suppose we perform many Pauli X measurements and the outcome of the measurement is always $+1$, then we can be certain that the state must be $\frac{1}{2}(|0\rangle + |1\rangle)$ which corresponds to a maximally coherent qubit. The larger the mean values of your measurement, the more confident you are that the state is close to the maximally coherent state, which in turn suggests that the state contains more coherence. Theorems 1 and 2 can be interpreted as a reflection of this confidence, generalized to arbitrary finite dimensional systems. The maximization over MIO and IO, which are both sets of operations that do not increase coherence, are further required in order to make this relationship more quantitative, such that it satisfies the axioms (C1)–(C3) (See section 2).

Furthermore, we also see that the chosen basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ is mutually unbiased with respect to the eigenbasis of σ_x, which again is the case being considered in proposition 1. Observe that the designation of the $+z$ direction is arbitrary. Given the direction $+x$, we can equivalently define any direction along the y–z plane as our new z axis. Consequently, we can be assured that any basis corresponding to a direction orthogonal to the x axis is mutually unbiased with respect to the eigenbasis of σ_x. A measurement along a given axis is therefore related to the amount of coherence along an orthogonal direction. More generally, any qubit observable M can always be written in the form $M = a_0\mathbb{1} + \vec{r}\cdot\vec{\sigma}$ where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the standard vector of Pauli matrices and $\vec{r}$ is a real three dimensional vector. In this case, the outcomes of the measurement M contains information about the coherence in a basis that is orthogonal to $\vec{r}$ in the Bloch sphere. Proposition 1 generalizes this observation to higher dimensions.

We now consider higher dimensional, multipartite scenarios and present numerical examples of our computable measure $\mathcal{C}_M^{\mathrm{MIO}}$.

Let us consider for spin systems the total magnetic moment operator. For a system of N spins we can choose as our classical basis $\bigotimes_{i=1}^N\{|\uparrow\rangle_i, |\downarrow\rangle_i\}$ where $\{|\uparrow\rangle_i, |\downarrow\rangle_i\}$ is the eigenbasis of the local spin-z operator. In order to witness the coherence between these basis states, a simple measurement of the magnetization in the x direction will suffice (see theorem 2 as well as proposition 1). The total spin-x operator is defined as

$$S_x = \sum_{i=1}^N S_x^i$$

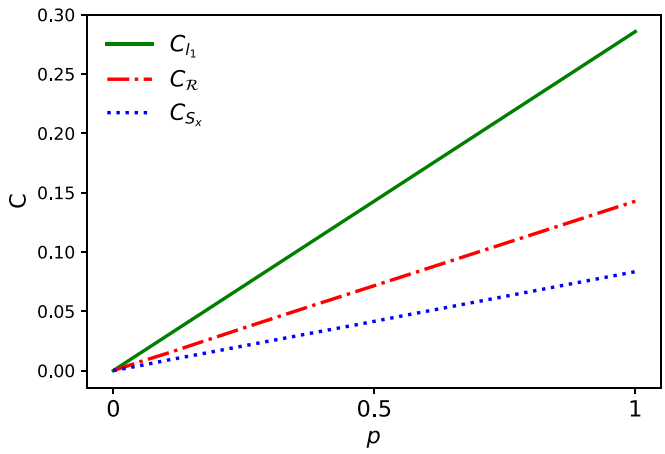

Figure 1. Comparisons among the l_1 norm of coherence $\mathcal{C}_{l_1}$(green, solid), the robustness of coherence $\mathcal{C}_{\mathcal{R}}$(red, dashed–dotted), and the coherence measure corresponding to magnetization measurement $\mathcal{C}_{S_x}$(blue, dotted). We consider the single parameter, 3 qubit state $\rho = (1 + p/7)\mathbb{1}/8 - p/7|w\rangle\langle w|$, where $|w\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ and $p \in [0, 1]$.

with local spin operators S_x^i. Choosing S_x as our observable, any measurement of $\langle S_x\rangle$ is automatically a lower bound to the corresponding coherence measure $\mathcal{C}_{S_x}^{\mathcal{O}}$. Note that because one can equivalently choose to measure the total magnetization along any direction on the equatorial plane, any non zero measurement of $\langle S_x\rangle$ directly implies the presence of coherence in the z direction.

One may also choose to find the 'optimal' measure by finding then implementing the optimal observable achieving $\mathrm{Tr}(W_\rho\rho) = \mathcal{C}_{\mathcal{R}}(\rho)$ [6, 7]. However, the physical implementation of such an observable W_ρ is not always simple. Instead, one may opt to perform a simpler measurement such as S_x, which corresponds to the computable measure $\mathcal{C}_{S_x}^{\mathrm{MIO}}$ using the semidefinite program described in theorem 3. This example illustrates how the resource requirements for experimentally detecting and measuring quantum coherence may be simplified via the direct application of theorem 2 and proposition 1. Figure 1 compares $\mathcal{C}_{l_1}$, $\mathcal{C}_{\mathcal{R}}$ and $\mathcal{C}_{S_x}^{\mathrm{MIO}}$ for the state $\rho = (1 + p/7)\mathbb{1}/8 - p/7|w\rangle\langle w|$ where $|w\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ and $p \in [0, 1]$. Note the hierarchy of the coherence measures $\mathcal{C}_M^{\mathrm{MIO}}(\rho) \leqslant \mathcal{N}_M\mathcal{C}_{\mathcal{R}}(\rho) \leqslant \mathcal{N}_M\mathcal{C}_{l_1}(\rho)$ (see theorem 4).

Several existing coherence measures can also be shown to fall under the framework that was discussed in this article. For instance, in [14], superradiance is studied within the context of coherence. In the idealized model for superradiance, there are N-number of two-level atomic systems with the energy levels denoted by $|e^{(i)}\rangle$ and $|g^{(i)}\rangle$ respectively. From this, we define the raising and lowering operators acting on the ith subsystem as $D_+^{(i)} := |e^{(i)}\rangle\langle g^{(i)}|$ and $D_-^{(i)} := |g^{(i)}\rangle\langle e^{(i)}|$, and the collective component of the emission rate, referred to as the superradiant quantity, is $\langle S_N\rangle = \sum_{i\neq j}\langle D_+^{(i)}D_-^{(j)}\rangle$. We see that S_N is a traceless observable whose leading diagonal elements are all zero in the axis defined by $|e^{(i)}\rangle$ and $|g^{(i)}\rangle$. This neatly falls underneath our framework, so any witnessing of superradiance is in fact, a witness of coherence between among basis states and a computable measure $\mathcal{C}_{S_N}^{\mathrm{MIO}}$ may be constructed. We note that this is a considerable improvement upon the original measure in [14], which uses the computationally difficult convex roof construction in order to generalize the measure to a general mixed state. A comparison of $\mathcal{C}_{S_N}^{\mathrm{MIO}}$ with other coherence measures for the pure state $|\psi(\theta)\rangle = (\cos(\theta)|g\rangle + \sin(\theta)|e\rangle)^{\otimes 3}$ is shown in figure 2. Note that in this case we chose $N = 5$ instead of $N = 3$ as the measure will saturate before the maximally coherent state is reached. Saturation of the measure is most easily explained using the following simplified example. We know that σ_x is a good choice of observable to measure the coherence w.r.t. the basis $\{|0\rangle, |1\rangle\}$. One may easily dilate this into a 4×4 block matrix of the form $\begin{pmatrix}\sigma_x & 0\\ 0 & 0\end{pmatrix}$. However, even though it is a 4×4 matrix, it is clear that the measure corresponding to this matrix will saturate with the superposition of only 2 orthogonal vectors via the state $\frac{1}{\sqrt{2}}(|0, 0\rangle + |0, 1\rangle)$. This coherence measure will therefore saturate before reaching the maximally coherent state $\frac{1}{2}(|0, 0\rangle + |0, 1\rangle + |1, 0\rangle + |1, 1\rangle)$, simply because the rank of the matrix is too low. A similar situation exists for $\mathcal{C}_{S_N}^{\mathrm{MIO}}$, which can be avoided by increasing the parameter N. Note that even when a measure saturates, this does not imply that the axioms (C1)–(C3) for coherence measures are violated.

Another example that falls under our framework is the fidelity of coherence distillation, which is discussed in greater detail in [34]. The fidelity of coherence is defined as the maximum overlap with the maximally coherent state optimized over some set of operations $\mathcal{O}$, which may be MIO or IO. Formally, it is defined as the following quantity:

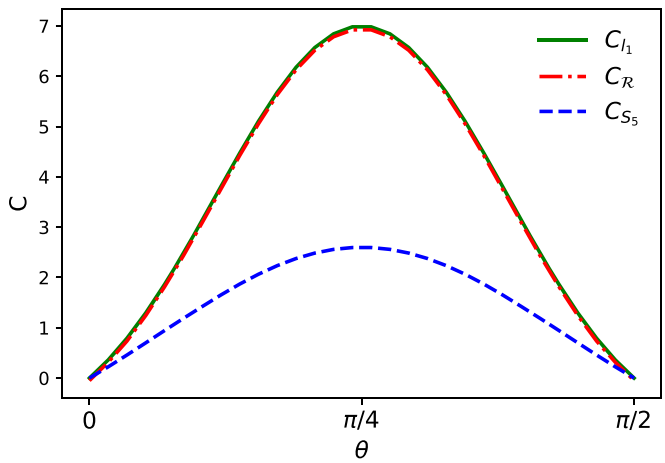

Figure 2. A comparison of $\mathcal{C}_{S_5}$(blue, dashed) with $\mathcal{C}_{l_1}$(green, solid) and $\mathcal{C}_{\mathcal{R}}$(red, dashed–dotted) for the state $|\psi(\theta)\rangle = (\cos(\theta)|g\rangle + \sin(\theta)|e\rangle)^{\otimes 3}\ |0\rangle^{\otimes 2}$. Note that a 5 qubit system (as opposed to a 3 qubit one) was chosen in order to avoid saturation of the measure $\mathcal{C}_{S_N}$ when $N = 3$. While the quantities are different, the qualitative behaviours are similar across $\theta \in \left[0, \frac{\pi}{2}\right]$. For pure states, $\mathcal{C}_{l_1}$ and $\mathcal{C}_{\mathcal{R}}$ matches in general.

$$F_{\mathcal{O}}(\rho) = \max_{\Phi \in \mathcal{O}} \mathrm{Tr}(\Phi(\rho)|\Psi\rangle\langle\Psi|),$$

where $|\Psi\rangle = \sum_{i=1}^{d} \frac{1}{\sqrt{d}}|i\rangle$ is the maximally coherent state. We see that the in the basis $\{|i\rangle\}$, the leading diagonal elements of the matrix $|\Psi\rangle\langle\Psi| - \frac{1}{d}\mathbb{1}$ are all zero, so $F_{\mathcal{O}} - \frac{1}{d}$ is in fact a coherence measure of the type described in theorem 2.

7. Conclusion

In this article, we demonstrated that every nontrivial Hermitian observable M corresponds to a coherence witness and a coherence measure $\mathcal{C}_M^{\mathcal{O}}$ for some specific incoherent bases, where the set of operations $\mathcal{O}$ may be either MIO or IO. In the case of MIO, we show that the measure is always computable via a semidefinite program, leading to an infinitely large set of computable coherence measures. The measures also show that the task of optimizing $\langle M\rangle$ is the same as the task of maximizing the coherence of the input state, up to the application of some incoherent operation (Theorem 2). They, therefore, have the operational interpretation of the usefulness of a given quantum state ρ for the purpose of optimizing the mean value of the observable $\langle M\rangle$. These coherence measures also satisfy a hierarchy $\mathcal{C}_M^{\mathrm{IO}}(\rho) \leqslant \mathcal{C}_M^{\mathrm{MIO}}(\rho) \leqslant \mathcal{N}_M\mathcal{C}_{\mathcal{R}}(\rho) \leqslant \mathcal{N}_M\mathcal{C}_{l_1}(\rho)$ (Theorem 4). This demonstrates that the robustness of coherence $\mathcal{C}_{\mathcal{R}}$ also has an interpretation as the ultimate usefulness of a state ρ for the purpose of optimizing *any* observable M. The l_1 norm of coherence $\mathcal{C}_{l_1}$ is also interesting because it is expressible in a closed form formula that is sometimes better suited for analysis, in comparison to $\mathcal{C}_M^{\mathrm{MIO}}$ and $\mathcal{C}_{\mathcal{R}}$ which both require numerical optimization to compute.

A key conclusion of our results, in particular theorem 2, is that every nontrivial quantum observable corresponds to a computable coherence measure, which also implies that every such quantum observable is also a coherence witness. This may in some cases allow coherence to be verified in the laboratory using simpler measurements, which was discussed in section 6. In spin systems, for example, a magnetization measurement is sufficient and relatively simpler to implement over a mathematically optimal measurement [8]. Moreover, the measurement outcomes of such observables are always, up to a constant displacement, a lower bound to a corresponding coherence measure $\mathcal{C}_M^{\mathcal{O}}$. Due to the hierarchy of coherence measures (see 4), they are also non-trivial lower bounds to the robustness of coherence and the l_1, although the upper bound given by $\mathcal{C}_M^{\mathcal{O}}$ is tighter in general. We hope that the techniques presented here will be useful to simplify the requirements for the detection and measurement of nonclassical quantum effects in the laboratory, as well as allow new coherence measures with novel physical interpretations to be discovered.

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