2008 Qualifying Exam: Quantum Mech.

Problem 1

Followings are wave functions of some states of the hydrogen atom labeled by the radial quantum number n:

$$\begin{aligned}
\Psi_{n,\ell,m}(r,\theta,\phi) &= \frac{1}{\sqrt{4\pi}}\left(\frac{1}{2a}\right)^{3/2}\left(2-\frac{r}{a}\right)e^{-r/2a} \\
\Psi_{n,\ell',m'}(r,\theta,\phi) &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\frac{1}{\sqrt{3}}\left(\frac{1}{2a}\right)^{3/2}\frac{r}{a}e^{-r/2a}. \quad (1)
\end{aligned}$$

Here, $a \equiv \hbar^2/me^2$ is the Bohr radius and $E_n = -\frac{e^2}{2a}\frac{1}{n^2}$.

(1) (10pts) From asymptotic behavior $r \to \infty$ of the wave functions and Hamiltonian, determine the principal quantum number n of the above states. By inspecting these wave functions, determine other quantum numbers ℓ, m, ℓ', m'.

(2) (10pts) Generate eigenfunctions of all other states carrying the same radial quantum number n.

(3) (15pts) The Hydrogen atom is now placed in an uniform external electric field E along the $z-$axis. In this situation, the perturbing Hamiltonian is

$$H' = -eEz = -eEr\cos\theta.$$

Using the first-order degenerate perturbation theory, calculate the energy shift.

(4) (10pts) Sketch qualitatively the energy levels as a function of the external electric field E. Indicate degeneracy or multiplicity if any levels are degenerate.

Useful Formulae :

$$L_x = i\hbar\left(\sin\phi\frac{\partial}{\partial\theta} + \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_y = i\hbar\left(-\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right)$$

$$L_\pm Y_{\ell m} \equiv (L_x \pm iL_y)Y_{\ell m} = \sqrt{(\ell\mp m)(\ell\pm m+1)}Y_{\ell m\pm 1}$$

$$\int d\cos\theta d\phi\, Y^\star_{\ell' m'}(\theta,\phi)\cos\theta Y_{\ell m}(\theta,\phi)$$

$$= \sqrt{\frac{(\ell+1)^2-m^2}{(2\ell+1)(2\ell+3)}}\delta_{\ell,\ell'-1}\delta_{m,m'} + \sqrt{\frac{\ell^2-m^2}{(2\ell+1)(2\ell-1)}}\delta_{\ell,\ell'+1}\delta_{m,m'}$$

$$\int_0^\infty dr\, r^n e^{-\beta r} = \frac{n!}{\beta^{n+1}}.$$

2008 Qualifying Exam: Quantum Mech.

Problem 2

Consider one-dimensional harmonic oscillator of spring constant k, charge e and mass m. You can introduce raising and lowering number operators via

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a); \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a^\dagger - a) \quad \text{where} \quad \omega \equiv \sqrt{\frac{k}{m}}.$$

(1) (`10 pts`) Express the Hamiltonian in terms of the number operators and derive the eigenvalue of each energy eigenstate.
(2) (`10pts`) Derive the root-mean-square distribution, $\sqrt{\langle x^2\rangle - \langle x\rangle^2}$, of each energy state.

Now the harmonic oscillator is perturbed by a uniform electric field E in $+x$ direction. Denote the perturbing part of the Hamiltonian as H'. At lower orders in perturbation theory, the energy and the wave function are given by

$$\begin{aligned} E_i &= E_i^0 + \langle i|H'|i\rangle + \sum_{n(\neq i)} \frac{|\langle i|H'|n\rangle|^2}{E_i^0 - E_n^0} + \cdots \\ \Psi_i &= |i\rangle + \sum_{n(\neq i)} \frac{|n\rangle\langle n|H'|i\rangle}{E_i^0 - E_n^0} + \cdots . \end{aligned}$$

where $|i\rangle$ and E_i^0 are the energy eigenstate and the eigenvalue of the unperturbed Hamiltonian, respectively. The ellipses denote higher-order corrections.

(3) (`10pts`) Calculate the energy of each energy level to second-order in the perturbation.
(4) (`10pts`) Calculate the induced electric dipole moment of each energy state.
(5) (`5pts`) Solve the corresponding classical problem: the harmonic oscillator of a particle of mass m, charge e in an external electric field E. Compare the classical result with the quantum mechanical result obtained in (4) using perturbation theory.

2008 Qualifying Exam: Quantum Mech.

Problem 3

Consider a charged particle moving in $(x,y)-$plane subject to a perpendicular magnetic field $B_z = B_0\theta(x)\theta(d-x)$. That is, the magnetic field is constant B_o in the strip region of width d and zero everywhere else. We will study the problem of scattering plane waves off such "magnetic barrier."

In general, the Hamiltonian for a charged particle coupled to a vector potential is obtained from Peierl's substitution of the momentum:

$$\mathbf{p} \to \left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)$$

Therefore, the Hamiltonian becomes

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e\mathbf{A}}{c}\right)^2.$$

(1) (7pts) Choose the gauge for the vector potential as $A_x(x,y) = A_z(x,y) = 0$ and $A_y(x,y) = 0$ for $x < 0$, then specify $A_y(x,y)$ for all x, y. Substitute your answer to the Schrödinger Hamiltonian of this problem:

$$H = \frac{1}{2m}\left[-\hbar^2\frac{\partial^2}{\partial^2 x} + \left(-i\hbar\frac{\partial}{\partial y} - \frac{eA_y}{c}\right)^2 - \hbar^2\frac{\partial^2}{\partial^2 z}\right]$$

and obtain explicit form of the Hamiltonian.

Consider now an electron incident from $x < 0$ and scattering off perpendicular to the magnetic barrier. For an incident wave $\exp(ikx)$, there will be in general a transmitted wave $T\exp(i\tilde{k}x)$ and a reflected wave $R\exp(-ikx)$.

(2) (8pts) The transmitted wave vector $\tilde{k}$ is determined by simple kinematics in terms of k and $B_0 d$. What is that relation?

(3) (10pts) For the magnetic barrier given, you will find that, below a certain critical energy E_0, $\tilde{k}$ is imaginary. What does this mean? Give a classical argument that leads to the same critical energy.

(4) (10pts) Find the reflection and transmission coefficients in the limit $d \to 0$ and $B_o \to \infty$ while holding $B_0 d$ fixed.

(5) (10pts) What is the direction of the transmitted probability flux? Note that it is *not* along the x-axis!

2008 Qualifying Exam: Quantum Mech.

Problem 4

Consider a particle of spin-1/2 in a uniform magnetic field $\mathbf{B}_0 = B_0\hat{z}$ ($B_0 > 0$). Its quantum dynamics is described by the Hamiltonian:

$$H = -\frac{ge}{2mc}\mathbf{S}\cdot\mathbf{B}.$$

Here, ge/mc is some positive constant and $\mathbf{S} = (\hbar/2)\boldsymbol{\sigma}$ is the spin operator expressed in terms of the Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

obeying the algebra $\sigma_i\sigma_j = \delta_{ij}\mathbb{I} + i\varepsilon_{ijk}\sigma_k$.

(1) (`10pts`) Take the Heisenberg representation for the spin operator, viz. $\mathbf{S}(t) = e^{iHt/\hbar}\mathbf{S}e^{-iHt/\hbar}$. Show that the expectation value $\langle\mathbf{S}(t)\rangle$ is given by

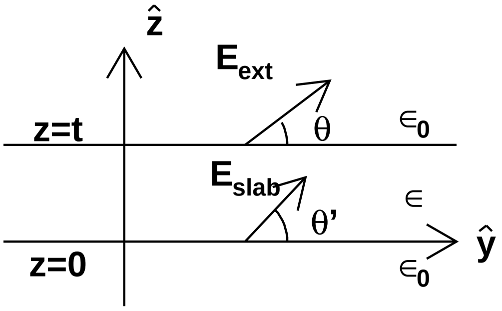
$$\frac{d\langle\mathbf{S}(t)\rangle}{dt} = \frac{\omega_0}{2}\langle\mathbf{S}(t)\rangle \times \hat{z}. \tag{2}$$

Also, find ω_0 and express it in terms of g, e, m, and B_0.
(2) (`10pts`) Solve Eq.(2) and obtain general solutions of $\langle S_x(t)\rangle$ and $\langle S_y(t)\rangle$.
(3) (`10pts`) Suppose that the initial state is $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$. Obtain the expectation values $\langle S_z(t)\rangle$ and $\langle S_x(t)\rangle$ for $t > 0$.
(4) (`15pts`) Assuming that the system was at its ground state for $t < 0$, i.e., $|\psi(t<0)\rangle = |+\rangle$ with $\langle\mathbf{S}(t)\rangle = +(\hbar/2)\hat{z}$, let us find out what happens when an additional time-dependent field $\mathbf{B}_1(t) = \hat{x}B_1\cos\omega_0 t$ is applied to the system on top of the field $\mathbf{B}_0 = B_0\hat{z}$. We shall assume that $\mathbf{B}_1(t)$ is applied during the period $0 < t < \tau_x$ and that $|B_1| \ll B_0$.
What value of τ_x will you choose should you want to get $\langle\mathbf{S}(\tau_x)\rangle = (\hbar/2)\hat{x}$, *i.e.*, $|\psi(t=\tau_x)\rangle = \frac{1}{\sqrt{2}}(|+\rangle + i|-\rangle)$?

2008 Qualifying Exam: Electrodynamics

Problem 1

An infinite insulating slab, with a finite thickness t, is inserted into a space which is filled with air. Dielectric constants of the slab and air are ε and ε_0, respectively. As shown in the figure, an external electric field $\mathbf{E_{ext}}$ with a magnitude of E_0 is applied at an angle of θ from the air-dielectric interfaces. The electric field $\mathbf{E_{slab}}$ inside the slab has a magnitude of E_0' and a direction with an angle of θ' from the interfaces. Please answer the following.



(1)(10 points) Write down the appropriate boundary conditions. Using these relations, derive E_0' in terms of $E_0, \theta, \varepsilon_0$ and ε.

(2)(15 points) What will be the relationship between θ and θ'?

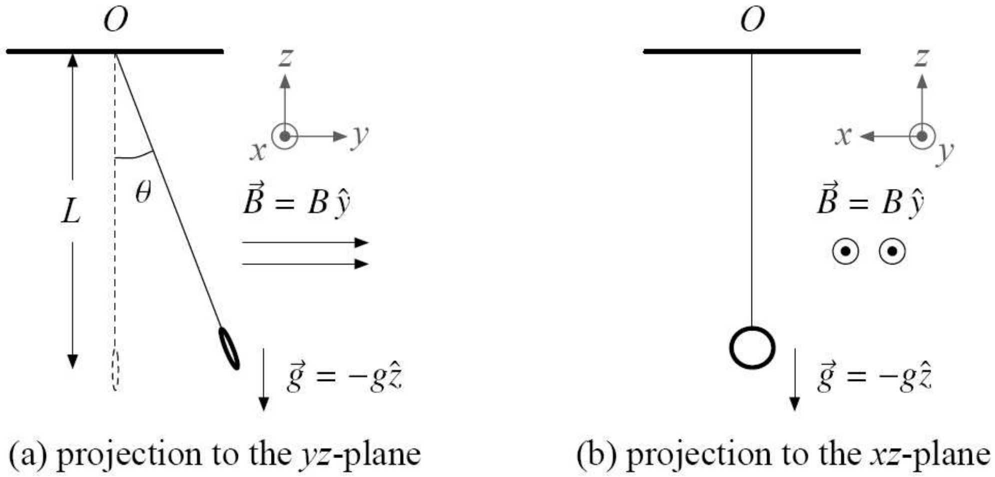
(3)(5 points) What is the induced surface charge density, σ, at the top interface?

(4)(10 points) Note that the induced surface charge densities at the top and bottom interfaces can generate electric field $\mathbf{E_d}$, which is called as the "depolarization electric field". Derive $\mathbf{E_d}$ for spaces inside and outside of the slab. What is a relationship among $\mathbf{E_d}, \mathbf{E_{ext}}$, and $\mathbf{E_{slab}}$ inside the slab?

2008 Qualifying Exam: Electrodynamics

Problem 2

Consider a circular ring made of a conducting wire. The area of the circle is A, the mass of the ring is m, and its total resistance is R. The ring is suspended at the end of a rigid, insulating rod of length L. Neglect the friction and the mass of the rod. Assume that the circle is sufficiently small ($\sqrt{A} \ll L$), so that the mass of the ring can be regarded as point-like. As the ring-rod system undergoes a pendulum motion, the rod is confined in the yz-plane, while the ring stays parallel to the x-axis (See the figure below). Finally, the pendulum is subject to a constant, uniform magnetic field $\mathbf{B} = B\hat{y}$ as well as the gravitional field $\mathbf{g} = -g\hat{z}$.



The ring-rod pendulum.

(1)(10 points) Suppose that the ring is in motion at an angle θ with an angular velocity $\omega = \dot{\theta} \equiv d\theta/dt$.Write down the induced electromotive force $\mathcal{E}$ in terms of θ and ω. Also, what is the electric current flowing around the ring?

(2)(10 points) Compute the magnetic moment μ, and the torque τ due to the interaction of μ with the background magnetic field $\mathbf{B}$.

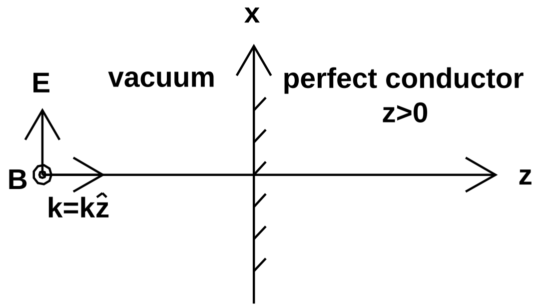
(3)(10 points) Write down the equation of motion for the pendulum, including both the magnetic force as well as the gravitational force.

(4)(10 points) Using the equation of motion and taking the electric current into account, show that the energy conservation law holds.

2008 Qualifying Exam: Electrodynamics

Problem 3

As shown in the following figure, a perfect conductor fills the half-space of $z > 0$. A plane wave with electric field, $\mathbf{E} = E_i \exp(ikz - iwt)\hat{x}$, is incident on the conductor from the left size. The space of $z < 0$ is in vacuum.



(1) (10 points) Find the electric and magnetic fields of the reflected wave.

(2) (10 points) What is the surface current density on the conductor? Write down its magnitude and direction.

(3) (10 points) Derive the charge density on the surface of the conductor.

(4) (10 points) What is the pressure on the conductor due to the wave?

2008 Qualifying Exam: Statistical Mechanics and Thermodynamics

<u>`Useful Formulae`</u>

$$\log N! \simeq N \log N - N$$

Problem 1

In the seminal year 1900, in order to explain the blackbody radiation spectrum, Max Planck proposed the following hypothesis: *For a given radiation frequency, there are N oscillators. Each oscillator radiates energy in the form of some number of finite "energy elements" each having energy* ε. For example, some oscillator has 3 energy elements with energy 3ε, other one with 5ε, etc.' This energy element is later referred as the energy quantum. If the total energy of N oscillators is U_N, there are then $M = U_N/\varepsilon$ energy elements distributed among oscillators. Planck succeeded in calculating the average radiating energy of an oscillator by obtaining the entropy of the system. Now, we would like to retrace Planck's logical steps.

(1) (`7pts`) Find total number W of different ways distributing M indistinguishable energy elements to N distinguishable oscillators. [**Hint**: *You may consider one-dimensional array of N cells and count the number of different ways to put M indistinguishable balls into the cells.*]

(2) (`7pts`) Assume that the numbers M and N are large enough. Show that the total entropy S expressed as a function of the average energy per oscillator, $U \equiv U_N/N = \varepsilon M/N$, is given by

$$S = Nk\left[\left(1+\frac{U}{\varepsilon}\right)\ln\left(1+\frac{U}{\varepsilon}\right) - \frac{U}{\varepsilon}\ln\frac{U}{\varepsilon}\right].$$

(3) (`8pts`) Using relations that can be derived by the first law of thermodynamics, express U as a function of the temperature T and ε. You may use the entropy expression in (2).

(4) (`8pts`) Planck's final expression for the energy distribution of the black body radiation is given as

$$E(\lambda) = \frac{8\pi ch}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1},$$

where $E(\lambda)d\lambda$ is the mean energy density of the black body radiation in the wavelength interval $[\lambda, \lambda + d\lambda]$. Derive this distribution.

2008 Qualifying Exam: Statistical Mechanics and Thermodynamics

Useful Formulae

$$\frac{\sinh x}{x} = 1 + \frac{x^2}{6} + \cdots; \quad \coth x - \frac{1}{x} = \frac{x}{3} + \cdots; \quad \int_{-\infty}^{+\infty} dx e^{-x^2/2\sigma} = \sqrt{2\pi\sigma}.$$

Problem 2

The canonical ensemble is widely used in equilibrium statistical mechanics. Here, the probability P_r for a system to be in a state r with energy E_r is given as

$$P_r = \frac{\exp(-\beta E_r)}{\sum_r \exp(-\beta E_r)}.$$

Here $\beta = 1/kT$, k being the Boltzmann constant. For the classical systems of N indistinguishable particles, the sum over states r actually means the integration over the phase space, $\int d\omega$, with

$$d\omega = \frac{1}{N! h^{fN}} \prod_{a=1}^{N} d\mathbf{p}_a d\mathbf{q}_a . \tag{1}$$

Here $d\mathbf{p}_a = dp_{1,a} dp_{2,a} \cdots dp_{f,a}$ and $d\mathbf{q}_a = dq_{1,a} dq_{2,a} \cdots dq_{f,a}$ are volumes of generalized momenta and generalized coordinates of the a-th particle with f being the degrees of freedom.

If the particle is a dumbbell shaped diatomic molecule with an electric dipole moment μ, the single particle Hamiltonian H in an external electric field E is written as

$$H = \frac{\mathbf{P}^2}{2m} + \frac{p_\theta^2}{2I} + \frac{p_\phi^2}{2I \sin^2\theta} - \mu E \cos\theta \tag{2}$$

where m is the mass, I is the principal moment of inertia of the molecule, $\mathbf{P} = (P_x, P_y, P_z)$ is the center of mass momentum, (θ, ϕ) are the spherical angles of the dipole $(0 < \theta < \pi, 0 < \phi < 2\pi)$, and (p_θ, p_ϕ) are the corresponding generalized momenta. The particles are confined in a cubic box of volume $V = L^3$.

(1) (7pts) Write down Eq.(1) explicitly for the case described in Eq.(2).

(2) (7pts) First consider the case when $E = 0$ in Eq.(2). Calculate the N particle partition function Q_N of the system with $E = 0$ neglecting the inter-particle interaction energy.

(3) (8pts) The system considered in (2) is heated from an initial temperature T_o to a final temperature $2T_o$ while holding the volume V fixed. Find the heat supplied to the system during this process.

(4) (8pts) Repeat the calculation of (3) for non-zero E but when $\mu E \ll kT_o$.

2008 Qualifying Exam: Mechanics

Problem 1

Consider a particle of mass m and electric charge q, moving in the $(x-y)$-plane under the influence of two types of forces: an attractive central harmonic force $-m\omega_o^2\mathbf{r} = -m\omega_o^2(x,y,0)$ and a force due to constant magnetic field $\mathbf{B} = (0,0,B)$ along z-direction perpendicular to the plane.

(1)(7 points) Denoting particle's position as $\mathbf{r}(t) = (x(t),y(t),0)$, show that the equation of motion is given by

$$m\ddot{\mathbf{r}}(t) = -m\omega_o^2\mathbf{r}(t) + q\dot{\mathbf{r}}(t)\times\mathbf{B}. \tag{1}$$

Extract equations of motion for $x(t)$ and $y(t)$, and for complex coordinate $z(t) \equiv x(t) + iy(t)$.

(2)(8 points) Consider first the situation with $\mathbf{B} = 0$. Solve the corresponding equation of motion from Eq.(1) explicitly and show that the orbit of the particle is in general an ellipse.

(3)(10 points) Consider next the situation with $\mathbf{B} \neq 0$. Solve the full equation of motion. When the magnitude of the magnetic field is relatively small, explain precisely what effect the applied magnetic field produces on the orbital motion found in (2).

(4)(5 points) Construct Lagrangian $\mathfrak{L}(x,y,\dot{x},\dot{y},\mathbf{A})$ of the system and check that Eq.(1) is derived from the Lagrangian. You may choose the gauge $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}\mathbf{r}\times\mathbf{B} = -\frac{B}{2}(y,-x,0)$. Is the Lagrangian invariant under the gauge transformation $\mathbf{A}(\mathbf{r}) \rightarrow \mathbf{A}(\mathbf{r}) + \nabla\Lambda(\mathbf{r})$? Are the equations of motion gauge invariant?

2008 Qualifying Exam: Mechanics

Problem 2

A pendulum of a massive body of mass m is hung over ceiling by a thin wire of length ℓ. Under surface gravity (g= surface gravity constant), the pendulum in equilibrium is in straight vertical position. Let θ denote angular displacement of the pendulum from the vertical position.

(1)(7 points) From equation of motion, show that $\theta(t)$ is the solution of the following equation:

$$\ddot{\theta}(t) = -\omega^2 \sin\theta(t) \qquad \text{where} \qquad \omega = \sqrt{\frac{g}{\ell}}. \tag{2}$$

(2)(7 points)If you want to devise a pendulum with period 1 second, how would you choose m and ℓ? Discuss whether the answer is reasonable.

(3)(8 points) The pendulum initially in equilibrium was kicked by an impulse P at right angle (i.e. horizontally). Using dimensional analysis, identify condition for P under which $\theta(t) \ll 1$ for all time t.

(4)(8points) Expand the right-hand side of Eq.(2) to the next-to-leading order assuming that $\theta \ll 1$. First find the leading order solution $\theta^0(t)$. Then one can compute iteratively to the next-to-leading order. Take an ansatz for the most general solution to next-to-leading order as

$$\theta^{(1)}(t) = \gamma \sin(\omega t) + \alpha t^\nu \cos(\omega t) + \beta \sin(3\omega t),$$

where α, β and γ are constants and $\nu \geq 1$. By solving iteratively and demanding the requisite initial condition, determine α, β, γ and ν.