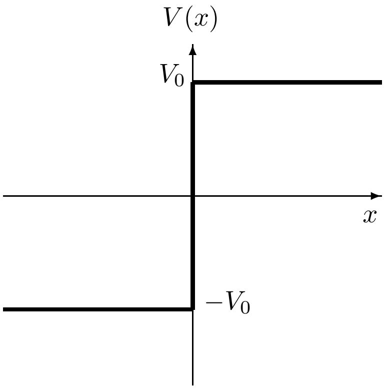
2009 년 1 월 6 일 시행 물리학부 석사과정 자격시험 문제

1 교시 양자역학

1. (25 points) A spin $1/2$ particle moves in a one dimensional potential $V(x)$ as below. The 2×2 Hamiltonian acting on a spinor $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ is

$$\mathcal{H} = \sigma_3 \frac{\lambda}{i} \frac{d}{dx} + \sigma_1 V(x)$$

where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and λ is a real constant.



a) (9 points) A plane wave of the form $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{ikx}$ is incident from the left. What is the amplitude ratio a_1/a_2 as a function of k?

b) (8 points) Find the complete wave function of Ψ(for all x) and the energy eigenvalue produced by this incident wave.

c) (8 points) Using the spin polarization $J = \Psi^\dagger \sigma_3 \Psi = |\Psi_1|^2 - |\Psi_2|^2$, compute the reflection coefficient R and the transmission coefficient T of the spin polarization for this barrier.

2. (25 points) Let $|j, m\rangle$ be a normalized eigenstate of angular momentum operators with following eigenvalues.

$$\langle j'm'|\mathbf{J}^2|jm\rangle = j(j+1)\hbar^2 \delta_{j'j}\delta_{m'm}$$
$$\langle j'm'|J_z|jm\rangle = m\hbar \delta_{j'j}\delta_{m'm}$$

a) (7 points) Show that $J_+|jm\rangle = C_{jm}|jm+1\rangle$ if $J_+ = J_x + iJ_y$, and calculate C_{jm} (up to phase).

b) (6 points) Write down the 3×3 matrix representation of J_x, J_y and J_z for $j = 1$.

c) (6 points) The rotation matrix can be parametrized by Euler angles as follows:

$$\begin{aligned} D^{(j)}_{m'm}(\alpha, \beta, \gamma) &= \langle jm'|e^{-\frac{iJ_z\alpha}{\hbar}} e^{-\frac{iJ_y\beta}{\hbar}} e^{-\frac{iJ_z\gamma}{\hbar}}|jm\rangle \\ &= e^{-i(\alpha m' + \gamma m)}\langle jm'|e^{-\frac{iJ_y\beta}{\hbar}}|jm\rangle \\ &= e^{-i(\alpha m' + \gamma m)} d^{(j)}_{m'm}(\beta). \end{aligned}$$

Show that $d^{(1)}(\beta)$ is given by

$$\begin{pmatrix} \frac{1}{2}(1+\cos\beta) & -\frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1-\cos\beta) \\ \frac{1}{\sqrt{2}}\sin\beta & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta \\ \frac{1}{2}(1-\cos\beta) & \frac{1}{\sqrt{2}}\sin\beta & \frac{1}{2}(1+\cos\beta) \end{pmatrix}$$

(Hint: $(J_y/\hbar)^3 = J_y/\hbar$ for $j = 1$)

d) (6 points) An atom is initially in a state with $j = 0$ and emits a photon of $j = 1$. Now the atom is in a state with $j = 1$. It can emit the second photon and goes back to the initial state with $j = 0$. What is the angular distribution of the second photon if the first photon is emitted to the z-direction?

3. (30 points) Consider an electron hopping in two quantum dots, where the lowest energy state of the electron in each quantum dot is ε_0 and the hopping amplitude is t. Let us assume the low energy states of this system can be described by the Hamiltonian:

$$\mathcal{H}_0|\phi_i\rangle = \varepsilon_0|\phi_i\rangle$$
$$\langle\phi_1|\mathcal{H}_0|\phi_2\rangle = \langle\phi_2|\mathcal{H}_0|\phi_1\rangle = -t$$

where i represents the site 1 or 2, and $t > 0$.

a) (8 points) Find the ground state energy E_0 and the corresponding eigenstate $|\Psi_0\rangle$ of $\mathcal{H}_0$.

b) (8 points) Now consider that an external magnetic field $\mathbf{B}_1 = B_0\hat{\mathbf{z}}$ is applied at the site 1 and $\mathbf{B}_2 = B_0(\cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}})$ at the site 2 with a constant B_0. We can write down the magnitude of the Hamiltonian as

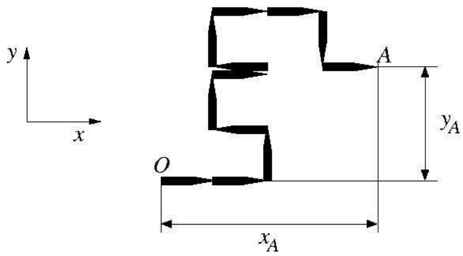
$$\mathcal{H}_1 = -\mu_0 \boldsymbol{\sigma}_1 \cdot \mathbf{B}_1 - \mu_0 \boldsymbol{\sigma}_2 \cdot \mathbf{B}_2.$$

Here μ_0 is a constant and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ represents the Pauli matrices. Using the perturbation theory in the weak field limit of $\mu_0 B_0 \ll t$, find the ground state energy and state vector (including both space and spin parts) as a function of θ.

c) (7 points) Now consider the strong field limit of $\mu_0 B_0 \gg t$ to find the ground state energy and state vector when the external magnetic field $\mathbf{B}_1 = B_0\hat{\mathbf{z}}$ is applied at the site 1 and $\mathbf{B}_2 = B_0\hat{\mathbf{z}}$ at the site 2.

d) (7 points) For the strong field limit of $\mu_0 B_0 \gg t$ as (c), find the ground state energy and state vector as a function of θ when the external magnetic field $\mathbf{B}_1 = B_0\hat{\mathbf{z}}$ is applied at the site 1 and $\mathbf{B}_2 = B_0(\cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}})$ at the site 2.

1. (20 points) We have a polymer which consists of N stick-shaped rigid monomer of length a. The polymer is on the plane and each monomer can be aligned only in four directions: up, down, left, and right, in two dimensions. One end of the polymer is fixed to the origin O, and the energy of the polymer is given by $E = \kappa(x_A + y_A)$. The polymer is in equilibrium at temperature T.



   a) (5 points) Calculate the partition function of the polymer.
   b) (5 points) Calculate the average energy U and the entropy S of the polymer.
   c) (6 points) Repeat the same calculation when each monomer can be aligned only in two directions, left and right, in one dimension.
   d) (4 points) Compare the results of (b) and (c) and explain the physical reasons for the difference.

2. (20 points) Let us consider a classical ideal gas composed of N indistinguishable monatomic particles contained in a box with volume V in three dimensions.

   a) (4 points) Obtain the equation of state.
   b) (4 points) We add the interaction between each pair of particles, and the Hamiltonian is given as

$$\mathcal{H} = \sum_i \left(\frac{p_i^2}{2m}\right) + \sum_{i<j} u_{ij}.$$

   Then, the equation of state may be written in the form

$$\frac{Pv}{k_B T} = \sum_{\ell=1}^{\infty} a_\ell(T) \left(\frac{\lambda^3}{v}\right)^{\ell-1}, \tag{1}$$

   where $v = V/N$ and $\lambda = h/(2\pi m k_B T)^{1/2}$. $a_\ell(T)$ are called the virial coefficients.
   The interacting potential $u(r)$ is given as

$$u(r) = \begin{cases} \infty & \text{for} \quad r < r_0, \\ -u_0 \left(\frac{r_0}{r}\right)^6 & \text{for} \quad r \geq r_0. \end{cases} \tag{2}$$

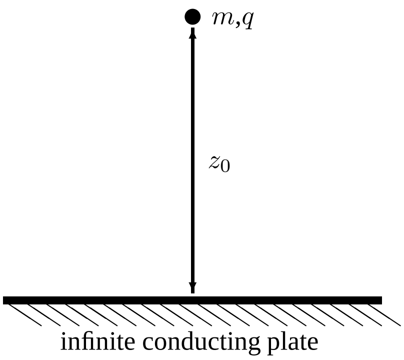
   When $(u_0/kT) \ll 1$, show that within the second order of virial expansion, the equation of state reduces to the van der Waals approximation,

$$(P + \frac{a}{v^2})(v - b) \approx k_B T. \tag{3}$$

   Derive the constants a and b explicitly.
   c) (4 points) We consider a gas-liquid phase transition governed by Eq.(3). Sketch the curve of P vs v for various temperature T, and find the critical temperature T_c, above which P decreases monotonically with v. Find the critical pressure P_c and volume v_c as well.
   d) (4 points) Near the critical temperature, show that $v_g - v_\ell \propto t^{1/2}$ when $t > 0$, where $t \equiv (1 - T/T_c)$ is assumed to be small, i.e., $t \ll 1$, v_g and v_ℓ are the volumes in gas and liquid phases, respectively.
   e) (4 points) Obtain the isothermal compressibility when $t < 0$ and $t > 0$, respectively.

3교시 전기역학

1. (30 points) A particle of mass m and charge q is released from rest from a distance z_0 above an infinite grounded conducting plate. Neglect relativistic effects and gravity.



   a) (7 points) Calculate the force on the charge and write down the equation of motion for the charge.

   b) (7 points) By integrating the equation of motion, obtain the time passed until the particle hits the plane. You can neglect the radiation loss and you may leave the answer in terms of a *dimensionless* integral.

   c) (9 points) The electric field from an accelerated charge is given as

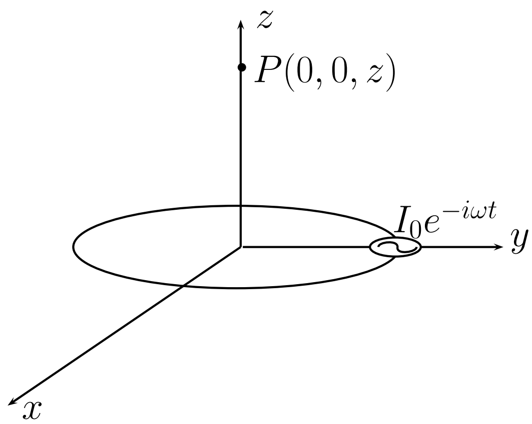
$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\partial^2}{c^2\partial t^2}\left[\hat{\mathbf{n}}\right]_{\text{ret}},$$

   in the limit of large distance from the source, where $\hat{\mathbf{n}}$ is a unit vector from the charge to the observation point. Using this expression, derive the radiated power as a function of z. (Hint: It might be easier to derive Larmor formula for the radiated power.)

Now let us replace the conducting plane by a semi-infinite dielectric ε. That is, for $z > 0$, there is a vacuum and for $z < 0$, the space is filled with the dielectric.

   d) (7 points) Calculate the force on the charge q when it is at a distance z_0 above the plane. (Hint: an image solution exists where image charges are placed at either $+z_0$ or $-z_0$.)

2. (25 points) An elliptical loop of semi-axes a and b lies in the xy-plane as shown in the following figure. The loop meets x-axis at $\pm a$ and y-axis at $\pm b$ and assume that $a > b$. An alternating current $I_0 e^{-i\omega t}$ is supplied to this loop. Assume the low frequency limit ($2\pi c/\omega$ is large compared to the dimensions of the problem.)



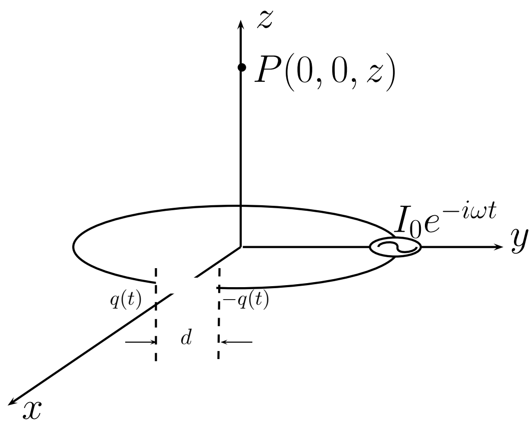
   a) (7 points) Using the Biot-Savart law, find the magnetic induction $\mathbf{B}(z)$ at a point $P(0,0,z)$ in the z-axis. You may leave *dimensionless* integral in your answer.

   b) (6 points) The magnetic induction due to a current loop can be expressed as that due to a magnetic dipole when $z \gg a, b$. Find the magnetic dipole moment of this current loop from the following definition:

$$\boldsymbol{\mu} = \frac{1}{2}\int \mathbf{x}' \times \mathbf{J}\, d^3x'. \tag{4}$$

   And show that the magnetic induction from (a) reduces to that of $\boldsymbol{\mu}$ when $z \gg a, b$.

Suppose that we cut the circuit around the $(a, 0, 0)$ point in the x-axis leaving a small gap of width d as shown in the following figure.



   c) (6 points) Find the induced charges at the gap from the continuity equation. (You can assume uniform current distribution inside the current loop.)

   d) (6 points) Find the magnetic induction $\mathbf{B}(z)$ at a point $P(0,0,z)$ in this case when $z \gg a, b$.

3. (25 points) Let us consider a scalar electromagnetic wave incident on a scattering center. The incident plane wave of

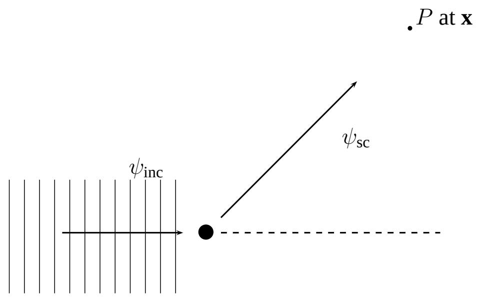
wavelength λ can be written as

$$\psi_{\text{inc}} = A_0 e^{ik\hat{\mathbf{n}}_0\cdot\mathbf{x}},$$

where $k = 2\pi/\lambda$ and $\hat{\mathbf{n}}_0$ is a unit vector to the direction of the propagation. When the scattering center is located at $\mathbf{x}'$, the scattered wave observed from a point P at $\mathbf{x}$ can be written as

$$\psi_{\text{sc}} = k^2 \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{r} f(\theta) L^3 \psi_{\text{inc}},$$

where θ is the scattering angle, L is a length scale for the scattering center.



a) (6 points) What is the differential cross section $\left(\frac{d\sigma}{d\Omega}\right)_0$ for a single scattering center?

b) (7 points) Now let us consider N scattering centers distributed at $\mathbf{x}' = \mathbf{x}_j$, $j = 1, \cdots, N$, localized around $\mathbf{x} = 0$. In the limit $L \ll \lambda \ll |\mathbf{x} - \mathbf{x}'|$, show that the differential cross section from N localized scattering centers can be written as

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{d\sigma}{d\Omega}\right)_0 |F(\mathbf{q})|^2 \\ \mathbf{q} &= k\hat{\mathbf{n}}_0 - k\hat{\mathbf{n}}, \end{aligned}$$

and find the expression for $F(\mathbf{q})$. (Hint: It might be easier to use an approximation $|\mathbf{x} - \mathbf{x}'| \simeq r - \hat{\mathbf{n}} \cdot \mathbf{x}'$, where $\hat{\mathbf{n}}$ is the unit vector in the direction to the point P at $\mathbf{x}$ from the origin.

c) (7 points) For a continuous distribution of scattering centers, $F(\mathbf{q})$ can be written as an integral such as

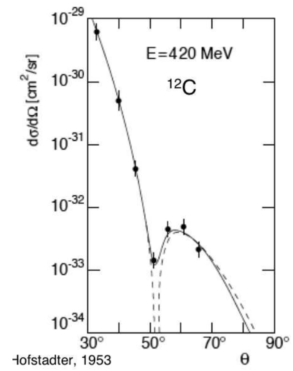
$$F(\mathbf{q}) = \int_V \rho(\mathbf{x}') e^{i\mathbf{q}\cdot\mathbf{x}'} d^3\mathbf{x}'.$$

If a sphere of radius R is uniformly filled with the N scattering centers, show that $F(\mathbf{q})$ is given as

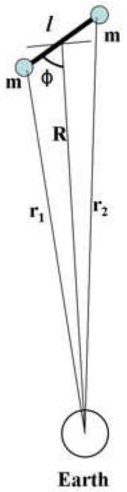
$$F(\mathbf{q}) = \frac{3N}{\alpha^3}(\sin\alpha - \alpha\cos\alpha),$$

with $\alpha = |\mathbf{q}|R$.

d) (5 points) When an electron beam of high energy is incident on a carbon foil, it can be considered electron matter wave scattered by charge distributions inside the carbon nucleus. R. Hofstadter, Nobel laureate in 1961, performed an experiment with an electron beam of energy 420 MeV and measured the following differential cross sections. From this measurement, the first diffraction minimum has been found at $|\mathbf{q}| \simeq 1.8\text{fm}^{-1}$ (fm is femto-meter, which is 10^{-15}m). Assuming that the carbon nucleus is a uniformly charged sphere and using the previous result, estimate the radius of the carbon nucleus. You can use the fact that $\tan 4.4934 = 4.4934$.

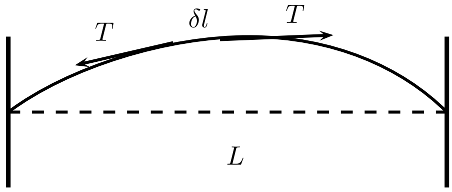


1. (20 points) A satellite under the gravitational pull is in a circular orbit of radius R. Consider a dumbbell-shaped satellite consisting of two point masses of mass m connected by a massless rod of length(l) much less than R (see figure). The orientation of the satellite relative to the direcrtion toward the center of the earth is measured by angle ϕ. (the mass of the earth is M, and the gravitational constant is G).



   a) (5 points) Obtain the Lagrangian.
   b) (3 points) Obtain the equation of motion for ϕ.
   c) (5 points) Determine the value of ϕ for the stable orientation of the satellite.
   d) (4 points) Show that the angular frequency of small angle oscillation of the satellite about its stable orientation is $\sqrt{3}$ times the orbital angular velocity of the satellite.
   e) (3 points) Explain briefly the physics of an automatic stabilization of the orientation of orbiting satellites.

2. (20 points) A uniform string has length L and mass per unit length ρ. It undergoes small transverse vibration in the xy-plane with its end poinds held fixed at $(0,0)$ and $(L,0)$ respectively. The tension is T. A velocity-dependent frictional force is present: if a small piece of length δl has transverse velocity v the frictional force is $-kv\delta l$.



Using appropriate approximations, the following equations hold for the vibration amplitude $y(x,t)$:

$$\begin{aligned}\frac{\partial^2 y}{\partial t^2} + a\frac{\partial y}{\partial t} &= b\frac{\partial^2 y}{\partial x^2} \quad \text{(i)}\\ y(0,t) &= 0 = y(L,t) \quad \text{(ii)}\end{aligned}$$

   a) (7 points) Find the constants a and b in (i). If you cannot do this part, take a and b as given positive constants and go on.
   b) (5 points) Find all solutions of (i) and (ii) which have the product form $y = X(x)\Omega(t)$. You may assume $a^2 < b/L^2$.
   c) (5 points) Suppose $y(x,0) = 0$, and

$$\dot{y}(x,0) = A\sin\left(\frac{3\pi x}{L}\right) + B\sin\left(\frac{5\pi x}{L}\right).$$

   Here A and B are constants. Find $y(x,t)$.
   d) (3 points) Suppose, instead, that $a = 0$ and $\dot{y}(x,0) = 0$ while

$$y(x,0) = \begin{cases} Ax & 0 \le x \le \frac{L}{2} \\ A(L-x) & \frac{L}{2} \le x \le L. \end{cases}$$

   Find $y(x,t)$.