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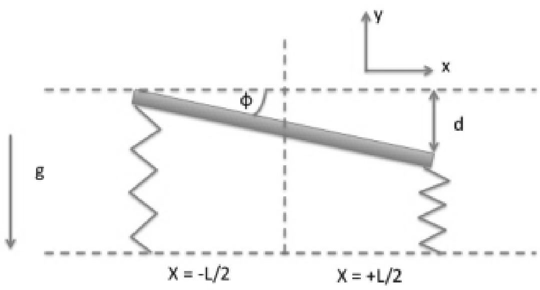
Subject : Classical Mechanics 2016. 6. 24.

1. (50 pts) Consider a spring with spring constant k and a length l.

    a) (7 pts) Find the Lagrangian $\mathcal{L}$ and the Lagrange's equation of motion if the spring oscillates as a simple harmonic oscillator with point particle (mass m). Find a solution of the equation of motion and a frequency of the oscillator.

    Now think a thin bar of length L and mass M with uniform density. This bar is supported by the spring at the corner shown in the Figure. The springs are confined so that they can move only vertically.

    b) (12 pts) Determine the Lagrangian for small amplitudes and the Lagrange's equation of motion.

    c) (13 pts) Find the normal modes of vibration and their frequencies.

    Extend this bar to a rectangular plate with same mass M (length and width are L). The spring supports the plate at each of corners. Same as above problems, the spring moves only in the vertical direction.

    d) (18 pts) Find the normal modes and frequencies for small amplitude oscillations (set up the perpendicular direction of the plate as y axis and transverse axes of the plate as x and z axes. Angle of rotation about the z-axis be θ).

소속대학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : Classical Mechanics 2016. 6. 24.

2. (50 pts) Consider a one dimensional chain of N identical balls of mass m connected by identical springs of spring constant K (See Figure 1). The equilibrium distance between the adjacent balls is a. The total length of the chain $L = Na$. The small displacements of the balls along the chain from the equilibrium positions are denoted by u^l.

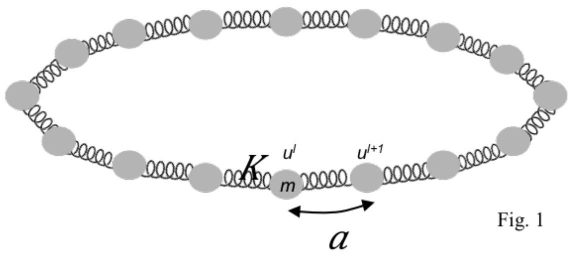


Fig. 1

a) (10 pts) Find the equation of motion for l^{th} ball and solve the equation of motion using a trial function $u^l = Ae^{i(kla-\omega t)}$ (Find the dispersion relation $\omega(k)$).

b) (5 pts) Assuming $a \ll L$, show that the equation of motion from (a) can be written as a wave equation.

c) (5 pts) Using the boundary condition, $u^N = u^0$, find the possible values of k and plot ω vs. k graph.

Now consider a one dimensional chain of balls in which balls of two different masses alternate (See Figure 2).

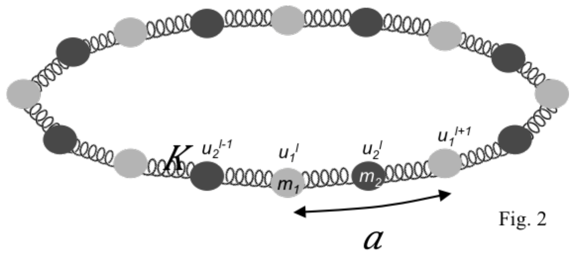


Fig. 2

d) (15 pts) Find the two simultaneous equations of motions for u_1^l and u_2^l. Sove the equation to find $\omega(k)$. (Hint: use trial function $u_j^l = A_j e^{i(kla-\omega t)}$.)

e) (8 pts) Find the expression of $\omega(k)$ and A_1, A_2 for small values of k.

f) (7 pts) Explain the difference of motions (modes) between the first case (all identical masses) and the second case (two different masses alternate) qualitatively (physically) using the answers to (a) to (e). ($\omega(k)$'s and the amplitudes A_1, A_2).

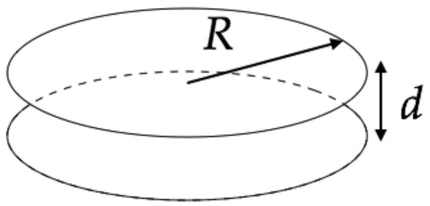
소속대학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : E&M 2016. 6. 24.

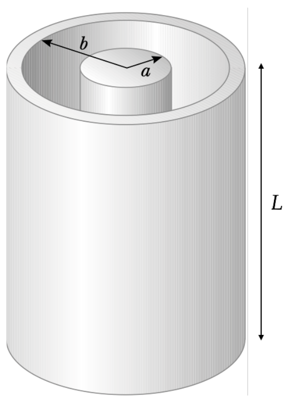
1. (50 pts) Let us consider capacitance of various forms of capacitors.

   a) Find out the capacitance of the following capacitors.

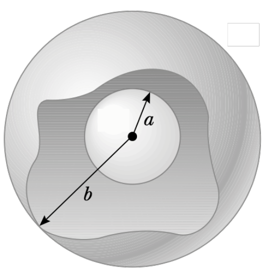
      i. (10 pts) parallel discs of radius R with a gap of distance d ($R \gg d$)

      ii. (10 pts) concentric cylinders with a radial gap of $d = b - a$ and length $L(\gg b)$.

      iii. (10 pts) concentric spherical shells with a radial gap of distance $d = b - a$

   b) (20 pts) Now suppose that we have a fixed amount (area) of conductors out of which one of the above capacitors can be made. Assuming that d is the same for all three shapes which shape will produce maximum capacitance? You can assume that there is no wasted material. Additionally the following conditions apply.

      i. For discs, $d \ll R$

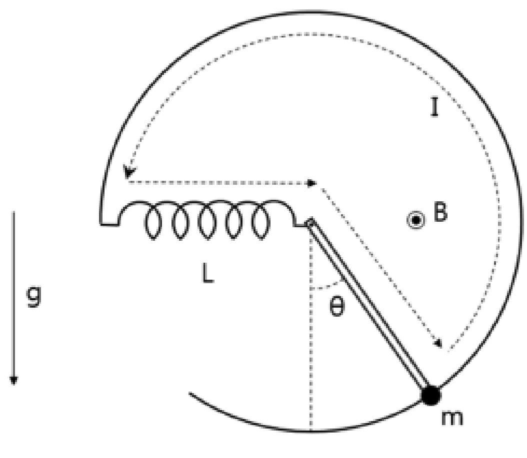
      ii. For cylinder, $d = b - a \ll a$ and $b \ll L$

      iii. For sphere, $d = b - a \ll a$

소속대 학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : E&M 2016. 6. 24.

2. (100 pts) An arc of conducting wire of radius r is connected to an inductor of self inductance L and a conducting rod, as shown in the figure. They form a circuit with negligible resistance, through which a current I flows. The rod is massless, and pivots around the origin. A bead of mass m is attached to the end of the rod, as shown in the figure. (Neglect all possible frictions between the rod/arc/bead/pivot.) Gravity is applied downwards in the figure, with gravitational acceleration g, and a uniform magnetic field B is applied, whose direction is shown in the figure.



a) (20 pts) Initially at $\theta(t=0)=0$, $I(t=0)=0$, one gives nonzero initial velocity to the bead. Using the Lenz's law, obtain the relation between $I(t)$ and $\theta(t)$. [Our convention is $I>0$ for counterclockwise current, as shown in the figure.]

b) (20 pts) If $\theta \ll 1$ (i.e. when the bead has sufficiently small initial speed), obtain the equation of motion for $\theta(t)$.

c) (20 pts) Combining the results of (a) and (b), show that the system makes a harmonic oscillation around $\theta = 0$, and compute the oscillation frequency.

d) (20 pts) Now consider the same system, at initial position $\theta(t=0)=\theta_0$ and initial current $I(t=0)=0$. Compute the relation between $I(t)$ and $\theta(t)$, as in problem (a), and compute the conserved energy.

e) (20 pts) Investigating the energy computed in problem (d), show that for $\theta_0 = \pi$ the system makes a harmonic oscillation for small enough initial speed, when B is larger than

$$B > B_* = \sqrt{\frac{4mgL}{r^3}}.$$

소속대학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : E&M 2016. 6. 24.

3. (50 pts) Let us consider a useful property called the mean value theorem (MVT).

   a) (20 pts) The MVT states that "for charge-free space, the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point." Let $\phi(\mathbf{r})$ is the electric potential defined on the given space. If we consider a sphere of radius a around the point, the average potential over the surface is defined as

$$\overline{\phi}_a = \frac{\oint_S \phi \, dA}{\oint_S dA},$$

   where the integral goes over the surface of the sphere and dA is the infinitesimal area element. Show that $\overline{\phi}_a$ is equal to ϕ at the center of the sphere. (Hint: try to consider $\overline{\phi}_a$ vs. $\overline{\phi}_{a+\delta a}$ for a small value of δa.)

   b) (15 pts) Consider a point charge q located at a distance z on the z-axis from the center of an imaginary spherical shell of radius a, such that $z > a$. Calculate the average potential on the surface of the shell and prove the MVT. (Hint: It might be easier to locate the origin of the coordinate system at the center of the spherical shell.)

   c) (15 pts) Using the MVT, prove the Earnshaw's theorem, "A charged particle cannot be held in a stable equilibrium by electrostatic forces alone."

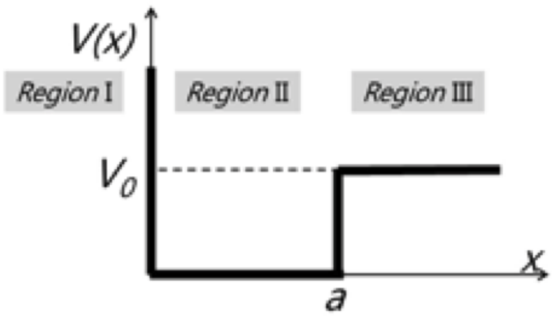
소속대 학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : Quantum Mechanics 2016. 6. 24.

1. (50 pts) Consider a particle with its mass m and energy E under the following one-dimensional potential,

$$V(x) = \begin{cases} +\infty & (x \leq 0) \\ 0 & (0 < x < a) \\ V_0 & (a \leq x) \end{cases}$$

with $V_0 > 0$.



a) (10 pts) Write down the Schrödinger equations for all regions and show that the particle cannot travel throughout the region III if $E < V_0$.

b) (15 pts) Write down the wave functions for all regions and their boundary conditions at $x = 0$ and $x = a$.

c) (15 pts) Find out the conditions of V_0 and a for the particle to have at least one bound state (Hint: you may use a graph to find out the number of possible bound states).

d) (10 pts) Show that the energy levels converge to those of an infinite potential well when V_0 becomes very large.

소속대 학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : Quantum Mechanics 2016. 6. 24.

2. (50 pts) Now consider a particle in a bound state under a general form of one-dimensional potential $U(x)$. This particle is spinless (spin=0). The potential $U(x)$ is not an explicit function of time t. As a result, the Hamiltonian $\widehat{H}$ is not an explicit function of time t, either.

   a) (10 pts) Suppose that we have a wave function $\psi(x,t)$ which satisfies the Schrödinger equation in (a). Using the Schrödinger equation and the properties of the wave function for the bound state at $x = \pm\infty$, show that the normalization of the wave function does not change over time, that is,

   $$\frac{d}{dt}\int_{-\infty}^{+\infty} \psi^*(x,t)\psi(x,t)\,dx = 0.$$

   b) (10 pts) Suppose that the particle is in an eigenstate of the Hamiltonian $\widehat{H}$ at $t = 0$ with eigenvalue E, that is

   $$\widehat{H}\psi(x,t=0) = E\psi(x,t=0).$$

   What would be the wave function $\psi(x,t)$ of the particle at $t > 0$? Express your answer in terms of $\psi(x,t=0)$, E and/or $\widehat{H}$.

   c) (10 pts) Show that the wave function $\psi(x,t)$ obtained in (c) is still an eigenstate of $\widehat{H}$.

   d) (10 pts) Suppose that you have two eigenstates of $\widehat{H}$,

   $$u_1(x) = \psi_1(x,t=0)$$
   $$u_2(x) = \psi_2(x,t=0)$$

   with the identical eigenvalue of E. Write down the 1-dimensional ordinary differential equation(s) satisfied by $u_1(x)$ or $u_2(x)$.

   e) (10 pts) Show that the two eigenstates $u_1(x)$ and $u_2(x)$ are linearly dependent, that is, show that $u_2(x) = cu_1(x)$ with some constant c. (Hint: Consider the Wronskian of the two functions u_1 and u_2, $W = u_2\frac{d}{dx}u_1 - u_1\frac{d}{dx}u_2$.)

소속대 학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : Quantum Mechanics 2016. 6. 24.

3. (50 pts) Consider two hydrogen atoms A and B. The energy of the 1s state is E_1. The electron in each hydrogen atom is in the 1s state. The two hydrogen atoms are brought together to form a hydrogen molecule. As the distance between the two decreases, there develops an overlap between the two states represented by the coupling $-\Delta = \langle G_A|\widehat{H}|G_B\rangle$ where $|G_A\rangle$ and $|G_B\rangle$ are 1s states of atoms A and B, respectively, and $\widehat{H}$ is the one electron Hamiltonian for the two hydrogen atoms (that is, two protons).

   a) (15 pts) Assuming that $|G_A\rangle$ and $|G_B\rangle$ are approximate eigen-states of $\widehat{H}$ with energy E_1, what is the 2×2 Hamiltonian matrix for the hydrogen molecule with $|G_A\rangle$ and $|G_B\rangle$ as the basis states?

   b) (10 pts) Find the eigen-states and energies for the hydrogen molecule.

   c) (10 pts) How much energy does the system lower by forming a molecule?

   d) (15 pts) Sketch the electron probability density for the two eigen-states along the line that goes through the nuclei of the two atoms.

4. (50 pts) For (a) to (c), suppose that we are living in a one-dimensional space and we have a particle of mass m (whose size is negligible) under a half-harmonic potential

$$V(x) = \begin{cases} \frac{m\omega^2}{2}x^2 & (x \geq 0) \\ +\infty & \text{otherwise.} \end{cases}$$

The Hamiltonian is given by

$$\widehat{H} = \frac{\widehat{p}_x^2}{2m} + V(x).$$

   a) (10 pts) Find (i) (4 pts) the period of classical oscillation of the system and (ii) (6 pts) the quantum mechanical energy eigenvalues. [Hint for (ii): The energy eigenvalues of a simple harmonic oscillator is $\left(n+\frac{1}{2}\right)\hbar\omega$ where $n = 0, 1, 2, \cdots$ Start from the eigenstates of a simple harmonic oscillator and find the ones which have vanishing probability amplitude at $x = 0$. Note that since the potential at $x < 0$ is infinite, the probability amplitude should vanish at $x < 0$.]

   b) (14 pts) Suppose that the potential at $x < 0$ is monotonically and very rapidly decreased in time from $t = 0$ so that it ends up being $V_0(x) = m\omega^2x^2/2$ at $t = \tau(> 0)$. (i) (10 pts) Now if the particle was in the first

소속대학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : Quantum Mechanics 2016. 6. 24.

excited state when $t < 0$, what will be the expectation value of $\widehat{p}_x^2/(2m) + V(x)$ at time $t \gg \tau$? (ii) (4 pts) Find a condition on τ for your answer to (i) to be valid.

c) (10 pts) Suppose that the Hamiltonian is

$$\widehat{H}(t) = \frac{\widehat{p}_x^2}{2m} + V_1(x,t)$$

for

$$V_1(x,t) = \frac{m\omega^2}{2}x^2 \times \begin{cases} 1 & (x \geq 0) \\ \left[1 + \frac{t}{\tau_1}\Theta(t)\right] & (x \leq 0) \end{cases}$$

where

$$\Theta(t) = \begin{cases} 1 & (t \geq 0) \\ 0 & (t < 0). \end{cases}$$

and $\tau_1 (> 0)$ is much longer than any characteristic time scale of the system. Initially at $t = 0$, the particle of mass m was in the first excited state of $\widehat{H}(t = 0)$. How much work has been done on the particle between $t = 0$ and $t = \infty$? (Hint: How will the number of nodes in the wavefunction evolve with time?)

d) (16 pts) Now consider a particle with mass m in a two-dimensional space whose Hamiltonian is given by

$$\widehat{H} = \frac{\widehat{p}_x^2 + \widehat{p}_y^2}{2m} + V_0(x,y)$$

where

$$V_0(x,y) = \frac{m\omega^2}{2}(x^2 + y^2).$$

(i) (8 pts) Find the energy eigenvalues of this system (4 pts) and the degeneracy of each energy level (4 pts). (Neglect spin.) (ii) (8 pts) Repeat this problem if $V_0(x,y)$ is replaced by

$$V(x,y) = \begin{cases} \frac{m\omega^2}{2}(x^2 + y^2) & (x \geq 0, y \geq 0) \\ +\infty & \text{otherwise.} \end{cases}$$

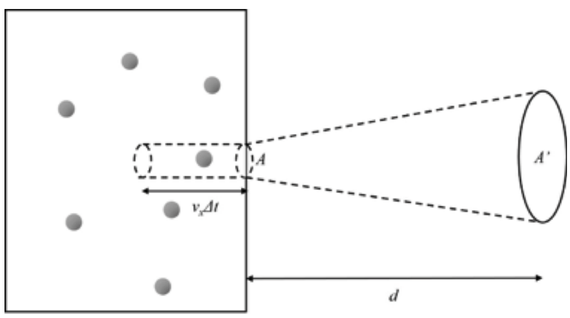
소속대학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : Thermal & Statistical Physics 2016. 6. 24.

1. (40 pts) Consider N molecules of monatomic ideal gas in a box of volume V in equilibrium at temperature T. The kinetic energy of a gas molecule is given by $mv^2/2$, where m is mass and v is the velocity. For a system in equilibrium at temperature T, the probability of a state with energy E is proportional to the Boltzmann factor, $\exp(-E/k_B T)$.

   a) (10 pts) Write the probability distribution for the velocity of particles inside the box so that the Maxwell-Boltzmann distribution is correctly normalized.

   Now suppose that we drill a small pinhole of area A on the box as shown in the Figure. Then gas molecules will start to escape from the box to a region of vacuum. Let us assume that the disturbance caused by the pinhole is small, and the gas molecules have a nearly equilibrium distribution inside the box.



   b) (10 pts) What is the average velocity of the gas molecules escaping from the box? (Suppose that the direction coming out of the hole is the $+x$ direction.)

   c) (10 pts) If a gas molecule with x-velocity component v_x was located within a cylinder of area A and length $v_x\Delta t$, it will emerge from the box through the pinhole. What is the rate that gas molecules escape from the box just after the hole is made on the box?

   d) (10 pts) Find the total number of gas molecules per unit time arriving at the detector with area A' located at a distance d from the pinhole.

Some useful integrals

$$\int_0^{+\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$
$$\int_0^{+\infty} xe^{-x^2}\,dx = \frac{1}{2}$$

소속대 학원	물리학부	학번		성명		감독교수 확 인	(인)

Subject : Thermal & Statistical Physics 2016. 6. 24.

2. (60 pts) Consider a two-dimensional square lattice system composed of Ising magnetic dipoles S_r, which can either point up ($S_\mathrm{r} = 1$) or point down ($S_\mathrm{r} = -1$). Here the subscript $\mathrm{r} = (x, y)$ denotes a lattice site where an Ising dipole is located, with integers x and y satisfying $1 \leq x \leq N$ and $1 \leq y \leq N$. Assuming that it is energetically favorable to have nearest-neighbor spins aligned in parallel, the sytem can be described by the following Hamiltonian,

$$H = -J \sum_{x=1}^{N} \sum_{y=1}^{N} \left[S_{(x,y)} S_{(x+1,y)} + S_{(x,y)} S_{(x,y+1)} \right], \quad J > 0,$$

where we assume a periodic boundary condition $S_{(x+N,y)} = S_{(x,y)} = S_{(x,y+N)}$.

a) (10 pts) Let us try to solve the previous Hamiltonian using a mean field approximation. We assume

$$\begin{aligned} S_\mathrm{r} S_{\mathrm{r}'} &\approx \langle S_\mathrm{r} \rangle S_{\mathrm{r}'} + S_\mathrm{r} \langle S_{\mathrm{r}'} \rangle - \langle S_\mathrm{r} \rangle \langle S_{\mathrm{r}'} \rangle \\ &= m(S_\mathrm{r} + S_{\mathrm{r}'}) - m^2, \end{aligned}$$

where $m = \frac{1}{N^2} \sum_{x=1}^{N} \sum_{y=1}^{N} \langle S_{(x,y)} \rangle$ is the average uniform magnetization. Show that the Hamiltonian can be approximated as

$$H \approx H_m = -4Jm \sum_{x=1}^{N} \sum_{y=1}^{N} S_{(x,y)} + 2JN^2 m^2.$$

b) (15 pts) Compute the partition function Z using H_m from (a) and the definition

$$\begin{aligned} Z &= \mathrm{Tr} \left[e^{-\frac{H_m}{k_B T}} \right] \\ &= \sum_{S_{(1,1)}=\pm 1} \cdots \sum_{S_{(N,N)}=\pm 1} e^{-\frac{H_m}{k_B T}}. \end{aligned}$$

c) (15 pts) Show that the average magnetization $m = \frac{1}{N^2} \sum_{x=1}^{N} \sum_{y=1}^{N} \langle S_{(x,y)} \rangle$ satisfies the following equation,

$$m = \tanh \left[\frac{4Jm}{k_B T} \right].$$

d) (10 pts) Using the result from (c), show that there is a critical temperature T_c which satisfies

$$m = \begin{cases} \text{nonzero}, & T < T_c \\ 0, & T > T_c \end{cases}$$

and find the expression of T_c.

e) (10 pts) Find m when m is very small by using the relation $\tanh x \approx x - x^3/3 + \ldots$.