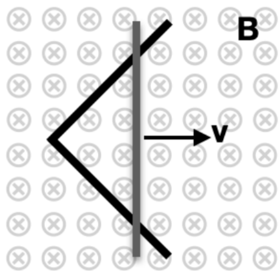
Physics II (Fall 2024): Final Exam Solution

Dec. 13, 2024

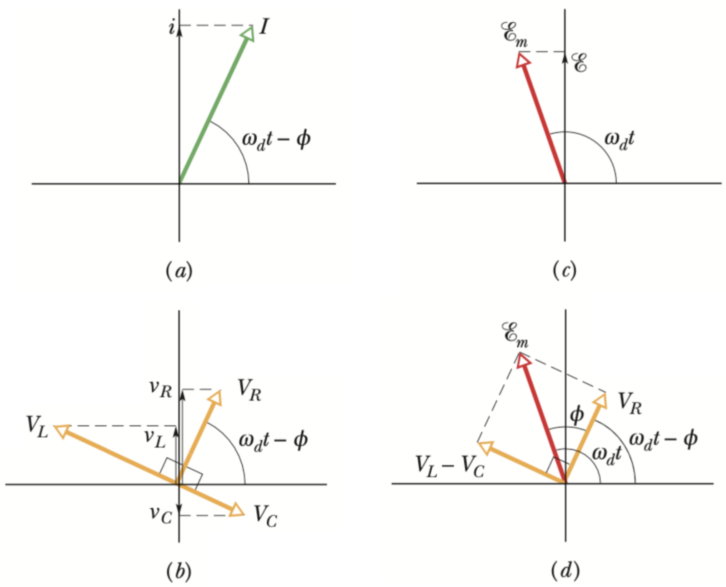
[total 25 pts, closed book, 90 minutes]

1. (a) [1 pt] In the figure below, two straight conducting rails form a right angle. A conducting bar in contact with the rails starts at the vertex[1] at time $t = 0$ and is pulled to the right with a constant speed of $v = 6.0\,\mathrm{m/s}$, while a magnetic field with $B = 0.35\,\mathrm{T}$ is directed into the page. Find the net emf $\mathcal{E}$ in the triangular loop formed by the rails and the bar at $t = 2.5\,\mathrm{s}$. What is the direction of the induced current in the loop?



(b) [2 pt] A resistor (resistance R), an inductor (inductance L), and a capacitor (capacitance C) are connected *in series* to an alternating emf, $\mathcal{E} = \mathcal{E}_\mathrm{m} \sin \omega_\mathrm{d} t$. The instantaneous voltages across R, L, and C are v_R, v_L, and v_C, respectively, while the currents through R, L, and C are i_R, i_L, and i_C, respectively. *(i)* Show that $\mathcal{E} = v_R + v_L + v_C$ and $i = i_R = i_L = i_C$ at any instant, where $i = I \sin(\omega_\mathrm{d} t - \phi)$ is the current through the emf. *(ii)* Using the phasor diagram below that was discussed extensively in the class, prove that $\mathcal{E}_\mathrm{m}^2 = V_R^2 + (V_L - V_C)^2$, where phasors V_R, V_L, and V_C represents v_R, v_L, and v_C, respectively. You must first explain the meanings of all phasors in figures (b) and (d) and how they are drawn. *(iii)* Demonstrate that you can also write $I = \frac{\mathcal{E}_\mathrm{m}}{Z}$ with $Z = \sqrt{R^2 + (\omega_\mathrm{d} L - 1/\omega_\mathrm{d} C)^2}$.

[1] vertex: 꼭짓점



(c) [2 pt] Now, a resistor, an inductor, and a capacitor are connected *in parallel* to an alternating emf, $\mathcal{E} = \mathcal{E}_\mathrm{m} \sin \omega_\mathrm{d} t$. *(i)* Show that $\mathcal{E} = v_R = v_L = v_C$ and $i = i_R + i_L + i_C$ at any instant, where $i = I \sin(\omega_\mathrm{d} t - \phi')$ is the current through the emf. *(ii)* Using *current* phasors I_R, I_L, and I_C, which represent i_R, i_L, and i_C, respectively, *on a current phasor diagram* indicate the phases of these three currents with respect to $\mathcal{E}$. Then, using this diagram, prove that $I^2 = I_R^2 + (I_C - I_L)^2$. *(iii)* Demonstrate that you can also write $I = \frac{\mathcal{E}_\mathrm{m}}{Z}$ with $\frac{1}{Z} = \sqrt{(1/R)^2 + (\omega_\mathrm{d} C - 1/\omega_\mathrm{d} L)^2}$.

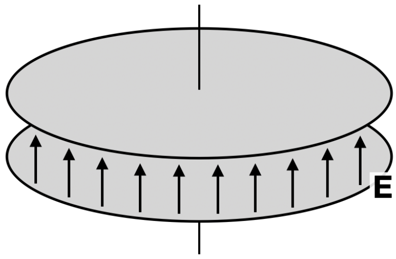
• (a) $\mathcal{E}_\mathrm{ind} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[B \cdot \frac{1}{2}(2vt)(vt)\right]$.

• (b) $\mathcal{E} = v_R + v_L + v_C$ means that phasor $\mathcal{E}_\mathrm{m}$ is equal to the vector sum of three voltage phasors, V_R, V_L, and V_C, yielding $\mathcal{E}_\mathrm{m}^2 = V_R^2 + (V_L - V_C)^2$. Then, $\mathcal{E}_\mathrm{m}^2 = (IR)^2 + (I\omega_\mathrm{d} L - I/\omega_\mathrm{d} C)^2 = I^2\left[R^2 + (\omega_\mathrm{d} L - 1/\omega_\mathrm{d} C)^2\right] = I^2 Z^2$. For details, see Chapter 31-4 of Halliday & Resnick.

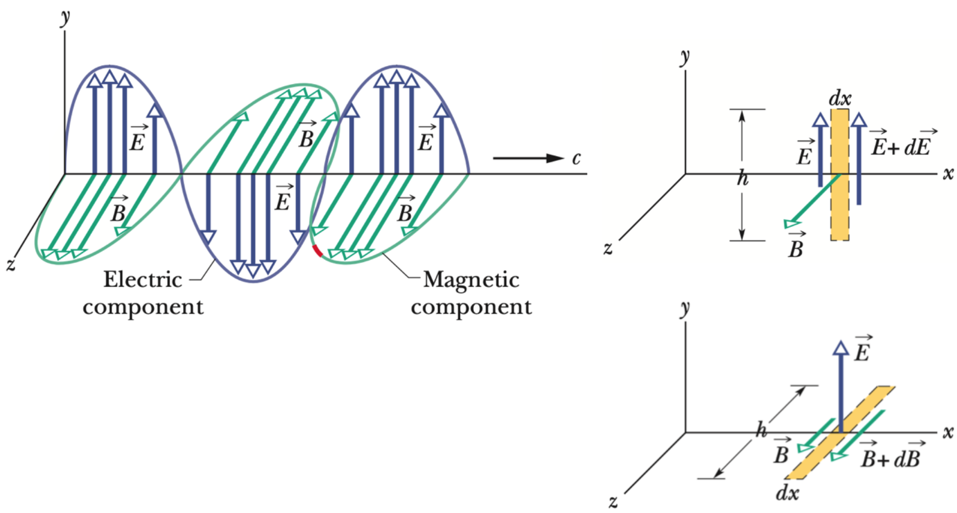
• (c) $i = i_R + i_L + i_C$ means that phasor I is equal to the vector sum of three *current* phasors, I_R, I_L, and I_C, yielding $I^2 = I_R^2 + (I_C - I_L)^2$. Then, $I^2 = (\mathcal{E}_\mathrm{m}/R)^2 + (\mathcal{E}_\mathrm{m}\omega_\mathrm{d} C - \mathcal{E}_\mathrm{m}/\omega_\mathrm{d} L)^2 = \mathcal{E}_\mathrm{m}^2\left[(1/R)^2 + (\omega_\mathrm{d} C - 1/\omega_\mathrm{d} L)^2\right] = \mathcal{E}_\mathrm{m}^2/Z^2$.

2. (a) [1 pt] Write down Maxwell's equations (in the integral form, *not* in the differential form), the four fundamental equations of electromagnetism.

(b) [2 pt] In the figure below, the magnitude of the electric field between the two circular parallel plates changes as $E(t) = (4.0 \times 10^5) - (6.0 \times 10^4)t$, where E is in V/m and t is in seconds. At $t = 0$, E is upward as indicated in the figure. The plate area is $A = 5.0 \times 10^{-2}\,\mathrm{m}^2$. *(i)* For $t > 0$, find the magnitude and direction (up or down) of the displacement current between the plates. *(ii)* What is the direction of the induced magnetic field (clockwise or counterclockwise when viewed from the top)?



(c) [2 pt] Starting from Faraday's law of induction and Maxwell's law of induction you wrote down in (a), derive the equation that describes a traveling plane electromagnetic wave. Here, you are asked to *explicitly* derive the wave equations for the electric field $E(x,t)$ and the magnetic field $B(x,t)$, and you are welcome to utilize the figure below that was discussed extensively in the class. (Note: Do not use the differential form of Maxwell's equations for this problem. You will have an opportunity to do so in other courses.)

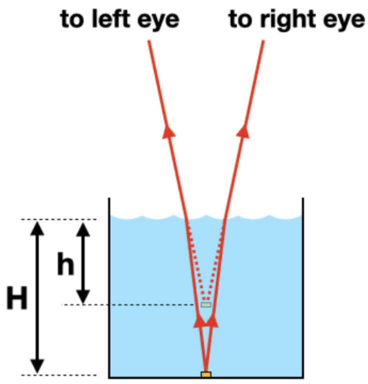


- (a) See Chapter 32-3 and Table 32-1 of Halliday & Resnick.

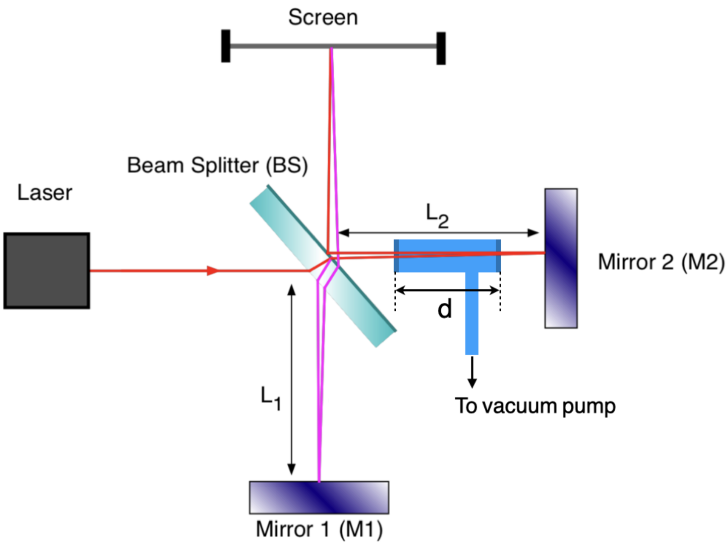
- (b) $i_d = \epsilon_0 \frac{d\Phi_E}{dt} = \epsilon_0 A \frac{dE}{dt} < 0$. The negative sign of i_d indicates that its direction is downward. This implies that the induced magnetic field is clockwise, which is obvious from Maxwell's law of induction, $\oint B \cdot ds = \mu_0 i_{d,\,\mathrm{enc}}$.

- (c) $\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \rightarrow hdE = -hdx\frac{dB}{dt} \rightarrow \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$, while $\oint \vec{B} \cdot d\vec{s} = \mu_0\epsilon_0 \frac{d\Phi_E}{dt} \rightarrow -hdB = \mu_0\epsilon_0 \left(hdx\frac{dE}{dt}\right) \rightarrow -\frac{\partial B}{\partial x} = \mu_0\epsilon_0\frac{\partial E}{\partial t}$. Then, by combining the two resulting equations and noting $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$, one can obtain $\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$ and $\frac{\partial^2 B}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}$. For details, see Chapter 33-1 of Halliday & Resnick.

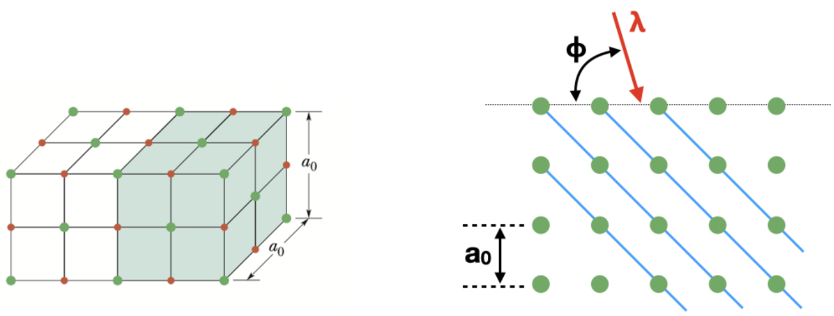
3. (a) [2 pt] A girl looks down at a coin lying at the bottom of a pool of water with depth H and refractive index n. Since she is looking with two eyes, she perceives the coin at the point where the extensions of the two intercepted rays meet, at apparent depth h (see figure). Assuming that the rays in the figure are close to a vertical axis through the coin, show that $h = H/n$. (Note: Because the rays are close to the vertical, you may utilize the small-angle approximation, $\sin\theta \approx \tan\theta \approx \theta$. The refractive index of air is 1.)



(b) [2 pt] In one of the two arms of a Michelson interferometer (L_2 in the figure below), an airtight chamber of length $d = 6.0\,\text{cm}$ is placed. The glass window at each end of the chamber has negligible thickness. He–Ne laser of wavelength $\lambda = 632.8\,\text{nm}$ is used to create the interference pattern; and then, evacuating the air from the chamber causes a shift of 55 bright fringes. Obtain the value of $(n-1)$ where n is the refractive index of the air in the laboratory.



(c) [1 pt] The regular array of atoms in a crystal can be regarded as a diffraction grating for short-wavelength waves such as X-rays. The first-order reflection (first-order diffraction maximum) from the reflecting planes, shown as blue lines below, occurs when an incident X-ray beam of wavelength $\lambda = 0.280\,\text{nm}$ makes an angle $\phi = 75°$ with the top surface of the crystal. What is the interplanar spacing d of the family of these reflecting planes, and the unit cell size a_0?



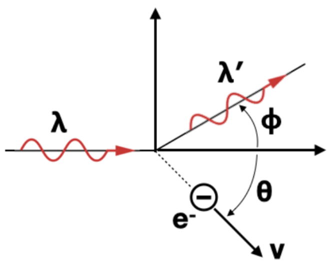
• (a) From Eq.(33-40) of Halliday & Resnick, $\sin\theta_1 = n\sin\theta_2$, where θ_1 is the angle of incidence and θ_2 is the angle of refraction. With small-angle approximation, $n = \frac{\sin\theta_1}{\sin\theta_2} = \frac{\tan\theta_1}{\tan\theta_2} = \frac{H}{h} \rightarrow h = H/n$.

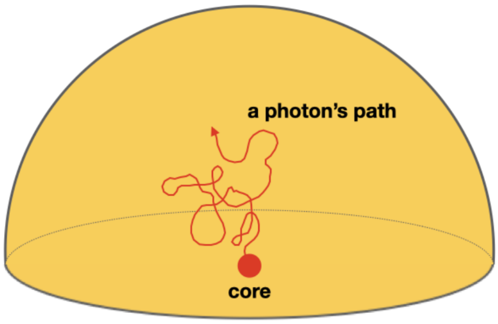
• (b) Similar to Eq.(35-43) of Halliday & Resnick, the phase change (in terms of wavelengths) between when the chamber is filled with air and is evacuated is $N_a - N_v = \frac{2d}{\lambda/n} - \frac{2d}{\lambda} = \frac{2d}{\lambda}(n-1)$ $\rightarrow \ n - 1 = \frac{\lambda}{2d}(N_a - N_v) = \frac{55\times 6.328\times 10^{-7}}{2\times 6.0\times 10^{-2}} = 2.9\times 10^{-4}$.

• (c) From Eq.(36-34) of Halliday & Resnick, $\lambda = 2d\sin\theta = \sqrt{2}a_0\sin(\phi - 45°)$.

4. (a) [2 pt] In this problem, we will compare the classical and relativistic expressions for kinetic energy, and see at what point we can no longer say that the two equations are approximately the same. *(i)* Write down the relativistic expression for kinetic energy K. Then, by applying the binomial theorem[2] to the Lorentz factor γ and retaining the first two terms of the expansion for K, show that the relativistic kinetic energy can be written as $K = \frac{1}{2}mv^2 +$ [first-order correction term]. *(ii)* Assume that the particle is an electron. If its speed v is $0.04c$, what is the value of the classical expression (i.e., $\frac{1}{2}mv^2$) and the first-order correction term? *(iii)* At what speed parameter β does the first-order correction become 10% or greater of the classical expression?

(b) [2 pt] Nuclear fusion reactions at the core of the Sun produce γ-ray photons with energies of about 1 MeV. In contrast, what we see emanating from the Sun's surface are photons of visible light. A simple model that explains this discrepancy is that a photon undergoes Compton scattering many times as it travels from the Sun's core to its surface. In fact, according to the models of the solar interior and the random walk theory,[3] a photon's average mean free path inside the Sun is about $l = 0.1$ mm, and it experiences about $N = 10^{26}$ such scattering events to travel from the Sun's core to its surface. Each scattering event deflects the photon's path from a straight line, thus lenghtening its travel time (see figure). *(i)* Using the values of l and N, estimate the time (in years) it takes a photon to travel from the Sun's core to its surface. (Note: Assume that the scattering is instantaneous. Given the nature of this problem, obtain your answers to this and the following questions accurate to just a *single* significant digit.) *(ii)* What is the wavelength of a γ-ray photon of energy 1 MeV? When this photon finally leaves the Sun, its wavelength is about 500 nm. What is the average increase in wavelength of this photon, $\Delta\lambda$, in a single Compton scattering event on its way out to the Sun's surface? *(iii)* Using the Compton shift formula, $\Delta\lambda = \frac{h}{m_e c}(1 - \cos\phi)$, find the angle through which the photon is deflected in this average scattering event. (Note: Since $\Delta\lambda$ is very small, you may use the small-angle approximation, $\cos\phi \approx 1 - \frac{1}{2}\phi^2$, where ϕ is in radians.)

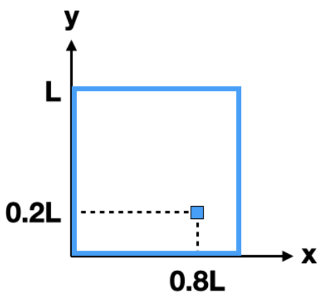




[2]binomial theorem: 이항정리, $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, where $\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$
[3]random walk theory: 무작위행보이론

- (a) With $\gamma = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-\frac{1}{2}} \simeq 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4$, the relativistic kinetic energy can be written as $K = (\gamma - 1)mc^2 \simeq mc^2\left[\frac{1}{2}\beta^2 + \frac{3}{8}\beta^4\right] = \frac{1}{2}mv^2 + \frac{3}{8}m\left(\frac{v^4}{c^2}\right)$.
- (b-1) $\frac{lN}{c} = \frac{0.1\,\text{mm}\times 10^{26}}{(3.0\times 10^8\,\text{m/s})\times(3.15\times 10^7\,\text{s/yr})} \simeq 1.1\times 10^6\,\text{yr}$.
- (b-2) $\Delta\lambda = \frac{1}{10^{26}}\cdot\left[500\,\text{nm} - \frac{(6.6\times 10^{-34}\,\text{J}\cdot\text{s})\times(3.0\times 10^8\,\text{m/s})}{(1.6\times 10^{-19}\,\text{J/eV})\times 1\,\text{MeV}}\right] \simeq \frac{1}{10^{26}}\cdot 500\,\text{nm} = 5.0\times 10^{-33}\,\text{m}$.
- (b-3) $\Delta\lambda \simeq \frac{h\phi^2}{2m_e c} \quad\rightarrow\quad \phi = \sqrt{\frac{2m_e c(\Delta\lambda)}{h}} = \sqrt{\frac{2\times 0.511\,\text{MeV}/(3.0\times 10^8\,\text{m/s})\times(5.0\times 10^{-33}\,\text{m})}{(6.6\times 10^{-34}\,\text{J}\cdot\text{s})/(1.6\times 10^{-19}\,\text{J/eV})}} = 6.4\times 10^{-11}$.

5. (a) [2 pt] A two-dimensional, infinite well of edge length $L = L_x = L_y = 120\,\text{pm}$ (see figure) contains an electron in its $E_{3,1}$ energy state — i.e., $(n_x, n_y) = (3,1)$. If a square probe of width of $d = 5.0\,\text{pm}$ is centered at $(x,y) = (0.8L,\, 0.2L)$, what is the probability of detection by the probe? (Note: To simplify the calculation, you may assume that d is much smaller than L.)



(b) [2 pt] The energy states of an electron within an atom can be split due to the interaction of the electron's spin and the magnetic fields *inside* the atom. Here we discuss two examples. (Note: For both questions, you are welcome to make use of the convenient constant introduced in the textbook, the Bohr magneton $\mu_B = \frac{e\hbar}{2m_e} = 9.3\times 10^{-24}\,\text{J/T} = 5.8\times 10^{-5}\,\text{eV/T}$.)

(i) First, we consider the famous yellow light from an excited sodium atom as it transitions from $(n,l) = (3,1)$ to $(n,l) = (3,0)$. The yellow light is actually composed of two closely spaced spectral lines called the sodium doublet, $\lambda_1 = 589.0\,\text{nm}$ and $\lambda_2 = 589.6\,\text{nm}$. This is because the excited state at $(n,l) = (3,1)$ is in fact split into two levels depending on whether the electron's spin magnetic moment $\vec{\mu}_s$ is parallel or antiparallel to the internal magnetic field $\vec{B}_1$ associated with *the electron's orbital motion.* Find the effective strength of this internal magnetic field, B_1.

(ii) Now we consider a hydrogen atom in its ground state. Just like an electron, a proton (a hydrogen nucleus) is a charged particle and has a spin magnetic dipole moment, so it produces a magnetic field around itself, which we call $\vec{B}_2$. The electron's ground state at $(n,l) = (1,0)$ is then split into two levels depending on whether the electron's spin magnetic moment $\vec{\mu}_s$ is parallel or antiparallel to the internal magnetic field $\vec{B}_2$ associated with *the proton's spin.* When an electron transitions between these two closely spaced energy levels, it can absorb or emit light of $\lambda_3 = 21\,\text{cm}$. Find the effective strength of this internal magnetic field, B_2.

• (a) From Eq.(39-19) of Halliday & Resnick, $\psi_{n_x,n_y}(x,y) = \sqrt{\frac{2}{L}}\sin\left(\frac{n_x\pi x}{L}\right)\sqrt{\frac{2}{L}}\sin\left(\frac{n_y\pi y}{L}\right) \rightarrow P(x,y) = |\psi(x,y)|^2 = \frac{4}{L^2}\sin^2\left(\frac{n_x\pi x}{L}\right)\sin^2\left(\frac{n_y\pi y}{L}\right)$. Therefore, the probability of detection at the probe's position is $P(0.8L, 0.2L)\cdot\Delta x\Delta y = \frac{4d^2}{L^2}\sin^2\left(\frac{3\pi\cdot 0.8L}{L}\right)\sin^2\left(\frac{\pi\cdot 0.2L}{L}\right) = 2.2\times 10^{-3}$.

• (b-1) From Eqs.(32-24), (32-27), (40-13) and (40-22) of Halliday & Resnick, $\Delta E = \frac{hc}{\lambda_1} - \frac{hc}{\lambda_2} = 2\vec{\mu}_s\cdot\vec{B}_1 = 2\mu_{s,z}B_1 = 2\mu_B B_1$, which gives $B_1 \simeq 18\,\mathrm{T}$.

• (b-2) $\Delta E = \frac{hc}{\lambda_3} = 2\vec{\mu}_s\cdot\vec{B}_2 = 2\mu_{s,z}B_2 = 2\mu_B B_2$, which gives $B_2 \simeq 5.1\times 10^{-2}\,\mathrm{T}$.

6. (a) [1 pt] Throughout the semester we discussed many examples in which concepts in classical physics are utilized in contemporary research or in explaining daily phenomena. In this regard, five of your peers presented their term projects at the end of the semester. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 3-4 sentences is expected to clearly convey the core physics idea of his/her term project. Use diagrams if desired. If you were one of the presenters, please choose someone else's.

(b) [1 pt] We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques such as order-of-magnitude estimation and/or dimensional analysis. Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem setup. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in $\sim$3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.

• (a) See the student presentation slides in Lecture 14-2 that include the collection of term project presentations by the students, and their video recordings on eTL.

• (b) See also the class slides for Lecture 12-2 that include many example problems, and the grading guideline.