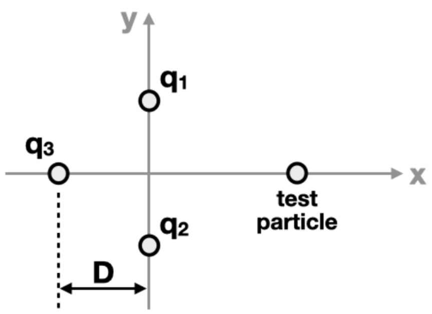
Physics II (Fall 2024): Midterm Exam Solution

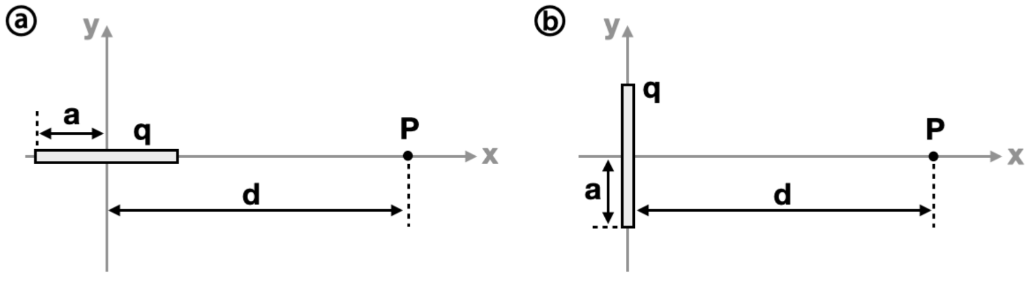
Oct. 25, 2024

[total 25 pts, closed book, 90 minutes]

1. (a) [2 pt] In the figure below, particles 1 and 2 are located at (0, 1.0 cm) and (0, −1.0 cm), respectively, with charges $q_1 = q_2 = -e$. Particle 3 is located on the negative x-axis with $q_3 = +5e$, and the test particle is at (2.0 cm, 0). *(i)* What is the distance D between the origin and q_3 if the net electrostatic force on the test particle due to the other particles is zero? *(ii)* If particles 1 and 2 were moved straight towards the origin but maintained their symmetry about the x-axis, would the required value of D be greater than, less than, or the same as in *(i)*?



(b) [2 pt] Positive charge q is uniformly distributed on a thin, nonconducting rod of length $2a$ along the x-axis between $[-a, +a]$ (case "a"), or along the y-axis between $[-a, +a]$ (case "b"; see figure). *(i)* For each case, find the magnitude and direction of the electric field at point P on the x-axis at distance d from the rod's midpoint. *(ii)* What form do your answers in *(i)* reduce to when P is very far from the charged rod, or if the rod is very short (i.e., $d \gg a$)? Explain why your answer is reasonable. *(iii)* What form does your answer for case "b" in *(i)* reduce to when P is very close to the charged rod, or if the rod is very long (i.e., $a \gg d$)?

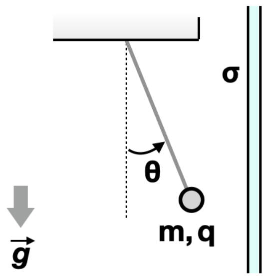


• (a) $-\frac{1}{4\pi\epsilon_0}\frac{e}{(1.0{+}4.0)\,\text{cm}^2}\times\frac{2}{\sqrt{5}}\times 2+\frac{1}{4\pi\epsilon_0}\frac{5e}{\{(2.0{+}D)\,\text{cm}\}^2}=0\ \rightarrow\ D=\sqrt{\frac{25\sqrt{5}}{4}}-2=1.7\,\text{cm}$. As you bring q_1 and q_2 closer to the origin, D must be decreased.

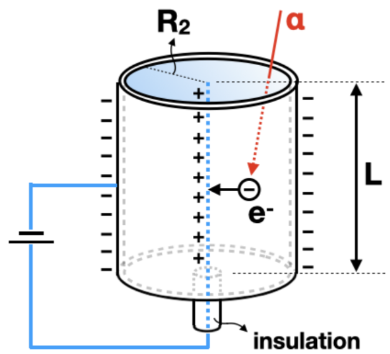
• (b-1) $E_{x,A}=\frac{1}{4\pi\epsilon_0}\int_{-a}^{+a}\frac{(q/2a)dx}{(d-x)^2}=\frac{1}{4\pi\epsilon_0}\frac{q}{d^2-a^2}$, and $E_{x,B}=\frac{1}{4\pi\epsilon_0}\int_{-a}^{+a}\frac{d}{\sqrt{d^2+y^2}}\cdot\frac{(q/2a)dy}{d^2+y^2}=\frac{1}{4\pi\epsilon_0}\frac{q}{d\sqrt{d^2+a^2}}$.

• (b-2) For $d\gg a$, $E_x\simeq\frac{1}{4\pi\epsilon_0}\frac{q}{d^2}$, while for $a\gg d$, $E_{x,B}=\frac{1}{4\pi\epsilon_0}\frac{q/a}{d\sqrt{(d/a)^2+1}}=\frac{1}{2\pi\epsilon_0}\frac{q/(2a)}{d\sqrt{(d/a)^2+1}}\simeq\frac{1}{2\pi\epsilon_0}\frac{q/(2a)}{d}$, which is equal to Eq.(23-12) of Halliday & Resnick.

2. (a) [1 pt] A small, nonconducting ball of mass $m=2.0\,\text{mg}$ and charge $q=+4.8\times10^{-8}\text{C}$ hangs from an insulating thread near a uniformly charged nonconducting sheet (shown in cross-section in the figure below; the sheet extends far vertically and into and out of the page). If the thread makes an angle $\theta=20°$ with the vertical, find the surface charge density σ of the sheet.



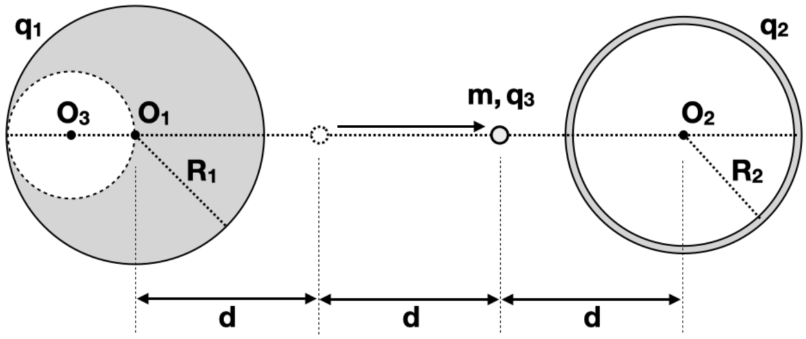
(b) [2 pt] A *Geiger counter* detects radiation such as α particles by using the fact that the radiation ionizes the air along its path. A thin, positively charged wire lies on the axis of a hollow, concentric cylindrical metal shell with an equal negative charge (see figure). A large potential difference is established between the wire and the shell, creating a strong electric field directed radially outward. When ionizing radiation enters the device, it ionizes a few air molecules. The resulting free electrons are accelerated towards the central wire by the electric field and become more energetic. As a result, they ionize many more air molecules on their way to the wire, producing even more electrons. The "avalanche" of electrons is collected by the wire, generating a current pulse and a clicking sound. Suppose that the radius of the central wire is $R_1=25\,\mu\text{m}$, the inner radius of the shell $R_2=1.5\,\text{cm}$, and the length of the shell $L=16\,\text{cm}$. If the electric field at the shell's inner wall is $E=2.4\times10^4\,\text{V/m}$, what is the total charge Q on the central wire? (Note: You may assume that L is much larger than R_2.)



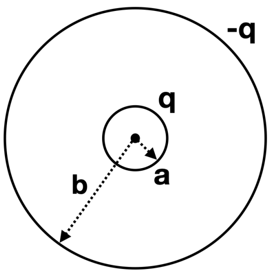
• (a) From Eq.(23-13) of Halliday & Resnick, $\frac{q|\sigma|}{2\epsilon_0}=mg\tan\theta$.

• (b) From Eqs.(23-7) and (23-12) of Halliday & Resnick, $Q=\epsilon_0E(2\pi R_2L)$.

3. (a) [2 pt] The figure below shows an arrangement of three objects. On the left, a nonconducting sphere of radius $R_1 = 0.7\,\text{cm}$ has been hollowed out so that the surface of the spherical hollow passes through the center of the sphere (O_1) and "touches" the left side of the sphere. On the right, a nonconducting spherical shell of radius $R_2 = 0.6\,\text{cm}$ is placed so that its center (O_2) lies on the straight line connecting O_1 and the center of the hollow (O_3; see figure). Charges $q_1 = +3.5 \times 10^{-9}\text{C}$ and $q_2 = -3.0 \times 10^{-9}\text{C}$ are uniformly distributed throughout the volume of the left and right object, respectively. O_1 and O_2, both fixed in space, are separated by $3d = 3.0\,\text{cm}$. Then, a particle of mass $m = 4.0\,\text{mg}$ and charge $q_3 = +2.0 \times 10^{-9}\text{C}$ is placed at a distance d from O_1, on a common line with all the centers on it. The particle starts from rest and moves along the common line. What is its speed v when it is $d = 1.0\,\text{cm}$ away from O_2?



(b) [2 pt] A metal sphere of radius a and charge q (> 0) is on an insulating stand at the center of a larger, spherical metal shell of radius b and charge $-q$ (see figure). Find the electric field as a function of distance r from the sphere's center, $\vec{E}(r)$, for all values of r ($0 < r < \infty$). Calculate and sketch $V(r)$, while taking $V(\infty) = 0$. What is the potential difference V_{ab} between the spheres? Finally, show that the capacitance of this two-sphere system is $C = 4\pi\epsilon_0 \left(\frac{ab}{b-a}\right)$.



(c) [1 pt] Using your findings in (b), verify that the following equality holds for the energy stored in the electric field between the spheres: $\frac{1}{2}CV_{ab}^2 = \int_{\mathcal{V}} \frac{1}{2}\epsilon_0 \{E(r)\}^2\, d\mathcal{V}$. (Note: In the class, we discussed this equality for the case of a parallel-plate capacitor. You may utilize $\int_{\mathcal{V}} f(r) d\mathcal{V} = \int_a^b f(r) 4\pi r^2 dr$ for a spherically symmetric function $f(r)$ and volume $\mathcal{V}$.)

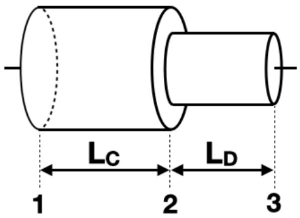
• (a) At the initial position of m, the electric potential is $V_i = \frac{1}{4\pi\epsilon_0}\left(\frac{(8q_1/7)}{d} + \frac{(-q_1/7)}{d+R_1/2} + \frac{q_2}{2d}\right) = 1904\,\text{V}$, while at its final position, $V_f = \frac{1}{4\pi\epsilon_0}\left(\frac{(8q_1/7)}{2d} + \frac{(-q_1/7)}{2d+R_1/2} + \frac{q_2}{d}\right) = -1084\,\text{V}$. Then, from Eq.(24-47) of Halliday & Resnick, $0 + q_3 V_i = \frac{1}{2}mv^2 + q_3 V_f \;\rightarrow\; v = \sqrt{\frac{2q_3(V_i - V_f)}{m}} = 1.7\,\text{m/s}$.

• (b) $\vec{E}_{(r<a)} = \vec{E}_{(r>b)} = 0$ while $\vec{E}_{(a<r<b)} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$. Integrating, $V_{(r>b)} = 0$, $V_{(a<r<b)} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$, and $V_{(r<a)} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$, which gives $V_{ab} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$. Meanwhile, $C =$

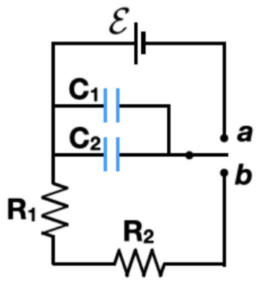
$4\pi\epsilon_0 \left(\frac{ab}{b-a}\right)$ can be found by following the steps that lead to Eq.(25-17) of Halliday & Resnick.

• (c) $\frac{1}{2}CV_{ab}^2 = \frac{1}{2} \cdot 4\pi\epsilon_0 \left(\frac{ab}{b-a}\right) \cdot \left[\frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)\right]^2 = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$, while $\int_{\mathcal{V}} \frac{1}{2}\epsilon_0 \{E(r)\}^2 d\mathcal{V} = \int_a^b \frac{1}{2}\epsilon_0 \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$.

4. (a) [2 pt] Wire C of length $L_C = 1.0\,\text{m}$ and wire D of length $L_D = 0.80\,\text{m}$ are joined as shown in the figure below, but are made from different materials. The resistivity and diameter of wire C are $2.0 \times 10^{-6}\Omega \cdot \text{m}$ and $1.0\,\text{mm}$, and those of wire D are $1.0 \times 10^{-6}\Omega \cdot \text{m}$ and $0.60\,\text{mm}$. If a current $I = 2.0\,\text{A}$ is set up in them, what is the electric potential difference between points 1 and 2, and between points 2 and 3? How about the rate at which energy is dissipated between points 1 and 2, and between points 2 and 3?



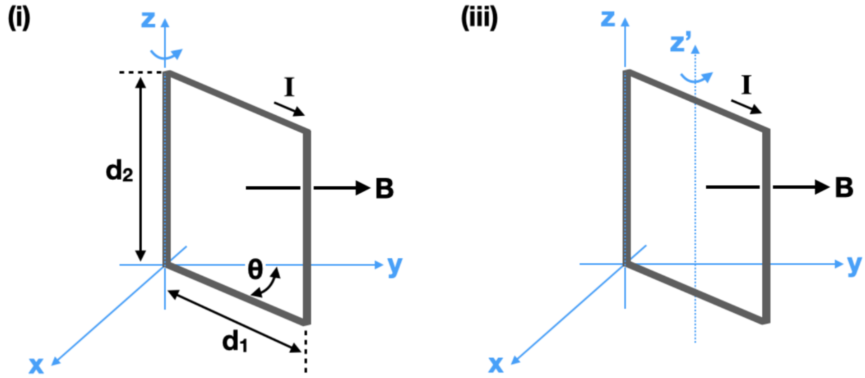
(b) [2 pt] In the circuit below, the resistances are $R_1 = 30\,\Omega$, $R_2 = 50\,\Omega$, and the capacitances are $C_1 = 15\,\mu\text{F}$, $C_2 = 20\,\mu\text{F}$, and the ideal battery has an emf of $\mathcal{E} = 45\,\text{V}$. First, the switch was held in position "a" for a long time; then, it was quickly thrown to position "b". *(i)* How long after the switch is moved to position "b" will the potential across C_1 be reduced to 10 V? *(ii)* What will be the current in the circuit at that time?



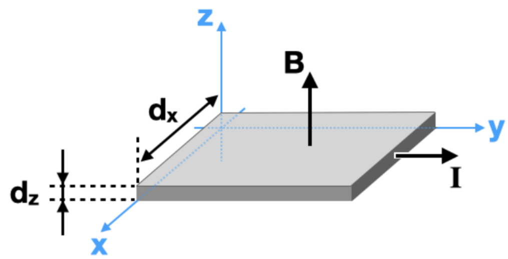
• (a) From Eq.(26-16) of Halliday & Resnick, the potential difference is $V_C = IR_C = I \cdot \rho_C \frac{L_C}{A_C}$, while the power dissipated is $P_C = I^2 R_C = I^2 \cdot \rho_C \frac{L_C}{A_C}$.

• (b) From Eqs.(27-39) and (27-40) of Halliday & Resnick, $V_C(t) = \frac{q(t)}{C} = \mathcal{E}e^{-t/RC}$ and $I(t) = \frac{dq(t)}{dt} = -\left(\frac{q_0}{RC}\right)e^{-t/RC} \;\rightarrow\; t = -RC\ln(V_C/\mathcal{E}) = 4.2\,\text{ms}$.

5. (a) [2 pt] The rectangular loop of $d_1 = 50\,\text{cm}$ and $d_2 = 70\,\text{cm}$ shown in the figure consists of 65 turns and carries a current of $I = 4.5\,\text{A}$. A uniform magnetic field of $B = 1.8\,\text{T}$ is directed along the $+y$-axis. The loop is pivoted about the z-axis (i.e., free to rotate about the z-axis) along one of the long sides. The angle θ shown in the figure is $30°$. *(i)* Determine the magnitude and direction of the net torque exerted on the loop (clockwise or counterclockwise when viewed from the top). (Note: You may want to be reminded that, when expressed in SI base units, $1\,\text{T} = 1\,\text{kg/A}\cdot\text{s}^2$.) *(ii)* State whether the angle θ will increase or decrease. *(iii)* If the loop were pivoted about an axis through the center of the loop, parallel to the z-axis (see figure), how would your answer in *(i)* change?



(b) [2 pt] A thin, rectangular silver strip of width $d_x = 1.2\,\text{cm}$ and thickness $d_z = 0.30\,\text{mm}$ carries a steady current $I = 120\,\text{A}$ in the $+y$-direction (see figure). The strip lies in a uniform magnetic field of $B = 1.0\,\text{T}$ along the $+z$-axis. The charge carrier density in silver is $n = 5.9 \times 10^{28}\,\text{electrons/m}^3$. *(i)* Find the magnitude of the electron drift velocity, v_d. *(ii)* Determine the magnitude and direction of the electric field due to the Hall effect. *(iii)* Prove that the Hall voltage (Hall potential difference) developed across the width of the strip is written as $V_\text{H} = \frac{BI}{ned_z}$. Find its numerical value. *(iv)* Discuss qualitatively how the Hall voltage would change if the material of the strip was a p-type semiconductor instead of silver.

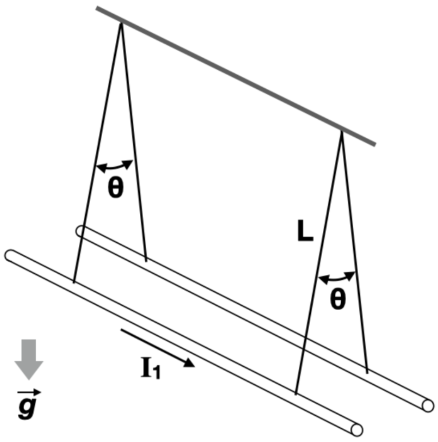


- (a) From Eqs.(28-33) or (28-37) of Halliday & Resnick, the torque on a magnetic dipole moment is $|\vec{\tau}| = |\vec{\mu} \times \vec{B}| = (NIA)B\sin(90° - \theta) = 160\,\text{N}\cdot\text{m}$.

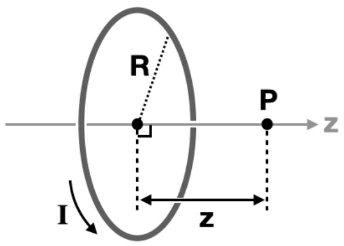
- (b-1) From Eq.(28-11) of Halliday & Resnick, the drift velocity of the electrons is $v_\text{d} = \frac{I}{neA} = \frac{I}{ned_xd_z} = \frac{120\,\text{A}}{5.9\times10^{28}/\text{m}^3\times1.6\times10^{-19}\,\text{C}\times3.6\times10^{-6}\text{m}^2} = 3.5\,\text{mm/s}$. The electric force due to the Hall effect must balance to the magnetic force; so, from Eq.(28-10) of Halliday & Resnick, $E = v_\text{d}B = 3.5\times10^{-3}\,\text{m/s}\times1.0\,\text{kg/A}\cdot\text{s}^2 = 3.5\times10^{-3}\,\text{kg}\cdot\text{m/s}^2\cdot\text{C} = 3.5\times10^{-3}\,\text{V/m}$.

- (b-2) Combining the equations above, one can show that $V_\text{H} = Ed_x = v_\text{d}Bd_x = \frac{I}{neA}\cdot Bd_x = \frac{BI}{ned_z} = 4.2\times10^{-5}\,\text{V}$. For details, see Chapter 28-3 and Eq.(28-12) of Halliday & Resnick.

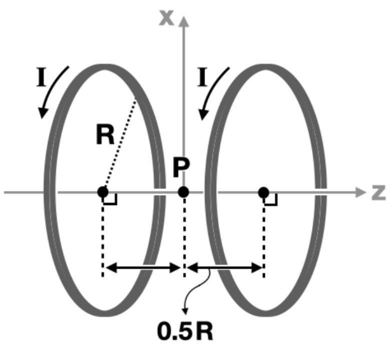
6. (a) [2 pt] Two long parallel wires are suspended from a common axis by four cables of length $L = 40\,\text{cm}$ (see figure). The wires have a mass per unit length of $\lambda = 0.013\,\text{kg/m}$, and one of the wires carries a current of $I_1 = 30\,\text{A}$. What is the magnitude and direction of the current I_2 in the other wire if the angle between the cables holding the two wires is $\theta = 12°$?

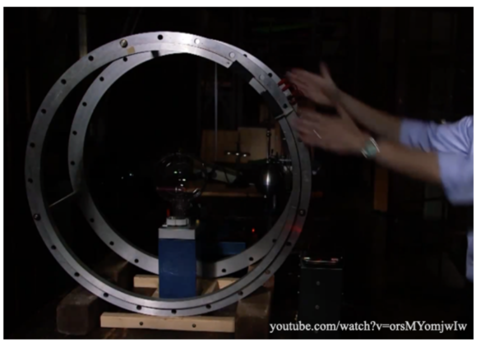


(b) [2 pt] Consider a magnetic dipole, that is, a single circular current loop of radius R and current I. Find the the magnetic field (magnitude and direction) at point P on the symmetry axis of the loop, at distance z from the center of the loop (see figure), by explicitly integrating the differential magnetic field, $\vec{B} = \int d\vec{B}$. Explain which expressions your answer reduces to *(i)* when P is at the center of the loop, and *(ii)* when P is very far from the loop ($z \gg R$).



(c) [1 pt] Now consider an arrangement known as a *Helmholtz coil*, which was briefly discussed in the class. It consists of two circular coaxial coils, each of N turns and radius R, separated by a distance R. The two coils carry equal currents I in the same direction. Find the the net magnetic field $\vec{B}$ (magnitude and direction) at the midpoint P between the coils. Demonstrate that the field is reasonably uniform near P ($z = 0$; see figure) by showing that its first and second derivatives are zero, i.e., $\frac{\partial \vec{B}}{\partial z}\Big|_{z=0} = \frac{\partial^2 \vec{B}}{\partial z^2}\Big|_{z=0} = 0$.





• (a) From Eq.(29-13) of Halliday & Resnick, the magnetic force acting on a small line element of the wire, Δl, is $F = \frac{\mu_0 (\Delta l)\cdot I_1 I_2}{2\pi\cdot 2L\sin(\theta/2)} = \lambda(\Delta l) g\tan(\theta/2)$.

• (b) $B_z = \int dB\sin\theta = \frac{\mu_0}{4\pi}\int_0^{2\pi}\frac{IR\cdot Rd\theta}{(z^2+R^2)^{3/2}} = \frac{\mu_0 IR^2}{2(z^2+R^2)^{3/2}}$ as in Eq.(29-26) of Halliday & Resnick. When $z = 0$, the equation reduces to Eq.(29-9) of Halliday & Resnick with $\phi = 2\pi$; when $z \gg R$, the equation becomes Eq.(29-27) of Halliday & Resnick.

• (c-1) $B_z = \frac{\mu_0 NIR^2}{2\{(z-R/2)^2+R^2\}^{3/2}} + \frac{\mu_0 NIR^2}{2\{(z+R/2)^2+R^2\}^{3/2}}$

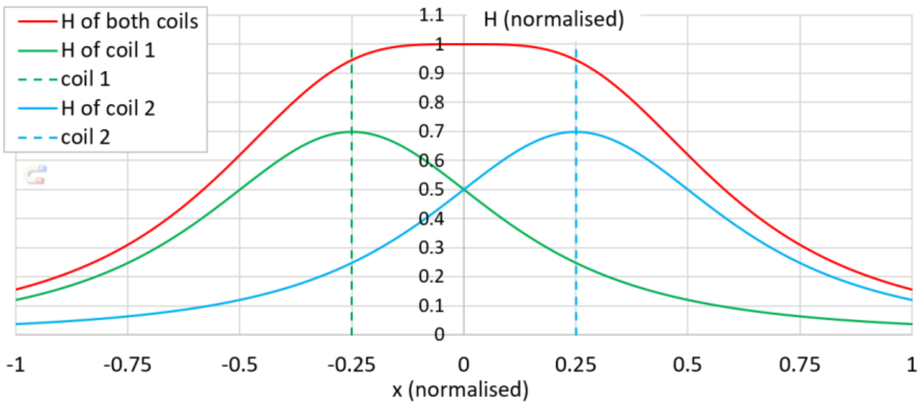
$$\rightarrow \frac{\partial B_z(z)}{\partial z} = -\frac{3\mu_0 NIR^2}{2}\left[(z-\tfrac{R}{2})\{(z-\tfrac{R}{2})^2+R^2\}^{-\frac{5}{2}} + (z+\tfrac{R}{2})\{(z+\tfrac{R}{2})^2+R^2\}^{-\frac{5}{2}}\right]$$

$$\rightarrow \frac{\partial^2 B_z(z)}{\partial z^2} = -\frac{3\mu_0 NIR^2}{2}\Big[\{(z-\tfrac{R}{2})^2+R^2\}^{-\frac{5}{2}} - 5(z-\tfrac{R}{2})^2\{(z-\tfrac{R}{2})^2+R^2\}^{-\frac{7}{2}}$$
$$+\{(z+\tfrac{R}{2})^2+R^2\}^{-\frac{5}{2}} - 5(z+\tfrac{R}{2})^2\{(z+\tfrac{R}{2})^2+R^2\}^{-\frac{7}{2}}\Big]$$

$\rightarrow$ Thus, $B_z(0) = \frac{\mu_0 NI}{R}\left(\frac{5}{4}\right)^{-\frac{3}{2}}$, $\left.\frac{\partial B_z(z)}{\partial z}\right|_{z=0} = 0$,

and lastly, $\left.\frac{\partial^2 B_z(z)}{\partial z^2}\right|_{z=0} = -\frac{3\mu_0 NIR^2}{2}\cdot 2\cdot\left[\left(\frac{5}{4}\right)^{-\frac{5}{2}}R^{-5} - 5\left(\frac{R^2}{4}\right)\left(\frac{5}{4}\right)^{-\frac{7}{2}}R^{-7}\right] = 0$.

• (c-2) Indeed, $B_z(z)$ has a plateau near $z = 0$, as seen in the plot below.



[image credit: Encyclopedia Magnetica (e-magnetica.pl/doku.php/helmholtz_coil)]