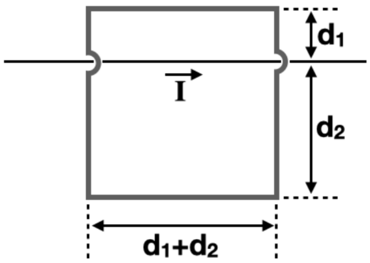
Physics II (Fall 2023): Final Exam Solution

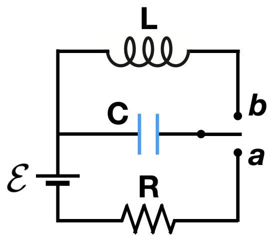
Dec. 8, 2023

[total 20 pts, closed book, 90 minutes]

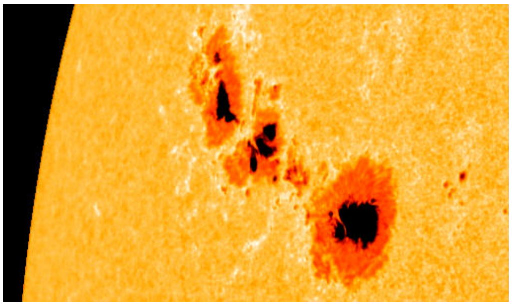
1. (a) [1 pt] In the wire arrangement shown below (with $d_1 = 4\,\mathrm{cm}$ and $d_2 = 12\,\mathrm{cm}$), the current in the straight wire changes as $I(t) = 5.0t^2 - 6.0t$, where I is in A (amperes) and t is in seconds. Find the net emf $\mathcal{E}$ at $t = 3.0\,\mathrm{s}$ in the square loop with resistance $R = 0.40\,\mathrm{m\Omega}$. What is the magnitude and direction of the induced current in the loop?



(b) [2 pt] In the circuit below, the resistance is $R = 14\,\Omega$, and the ideal battery has an emf of $\mathcal{E} = 35\,\mathrm{V}$. The capacitor of $C = 6.2\,\mu\mathrm{F}$ is initially uncharged, and the inductor has $L = 54\,\mathrm{mH}$. First, the switch was moved to position "a" and held there for a long time; then, it was quickly thrown to position "b". What are the angular frequency ω and the current amplitude I_m of the resulting oscillations?



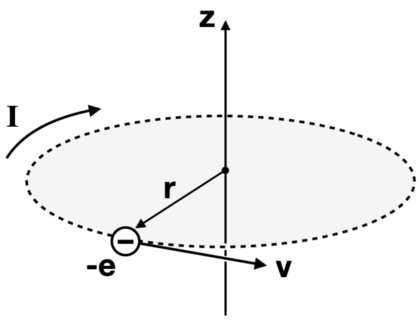
(c) [1 pt] *Sunspots* appear temporarily on the solar photosphere as dark patches of sizes up to 25,000 km in radius. Their surface temperature is reduced by concentrations of magnetic flux which prohibit convection, making them appear darker than their surrounding areas. Magnetic fields within a sunspot can be as strong as $B = 0.40\,\mathrm{T}$ while those on the Earth are only tens of μT. Let us assume that the material in a sunspot has a density of $\rho = 3.0 \times 10^{-4}\,\mathrm{kg/m^3}$, and that its permeability is μ_0. If 100% of the magnetic field energy stored in a sunspot is used to eject the sunspot's material away from the Sun's surface, what would be the ejection velocity v? Check if this magnetic field energy alone is enough to make the material escape the Sun's gravity, whose mass and radius are $M_\odot = 2.0 \times 10^{30}\,\mathrm{kg}$ and $R_\odot = 7.0 \times 10^5\,\mathrm{km}$, respectively. (Note: You may want to be reminded that, when expressed in SI base units, $1\,\mathrm{T} = 1\,\mathrm{kg/A \cdot s^2}$.)



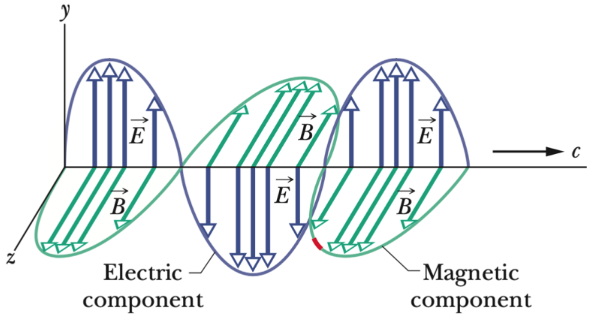
- (a) $\mathcal{E}_{\rm ind} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\left[\int_{d_1}^{d_2} \frac{\mu_0 I(t)}{2\pi r} \cdot (d_1 + d_2) dr\right] = -\frac{\mu_0(d_1+d_2)}{2\pi} \ln\left(\frac{d_2}{d_1}\right) \cdot \frac{dI}{dt}$.
- (b) $I_{\rm m} = \texttt{max}\left(\frac{dQ}{dt}\right) = \omega Q = \frac{1}{\sqrt{LC}} \cdot CV$.
- (c) From Eq.(30-55) of Halliday & Resnick $u_B = \frac{B^2}{2\mu_0} = \frac{1}{2}\rho v^2 \;\rightarrow\; v = \sqrt{\frac{B^2}{\rho\mu_0}} = 2.0 \times 10^4\,\mathrm{m/s}$.

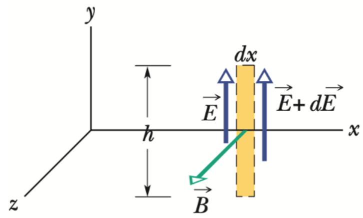
Meanwhile, from Eq.(13-28) of Halliday & Resnick, for the Sun $v_{\rm esc} = \sqrt{\frac{2GM_\odot}{R_\odot}} = 6.2 \times 10^5\,\mathrm{m/s}$.

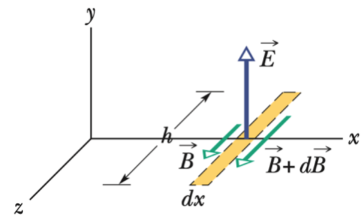
2. (a) [1 pt] For an electron moving along a circular path with radius r that is much larger than an atomic radius, show that its orbital magnetic dipole moment $\vec{\mu}$ can be written as $\vec{\mu} = -\frac{e}{2m}\vec{L}$ where m is the mass of an electron, and $\vec{L}$ is its orbital angular momentum. It should be noted that the derivation does not apply to an electron within an atom, for which quantum theory is required.



(b) [2 pt] Write down Maxwell's equations (in the integral form, *not* in the differential form), the four fundamental equations of electromagnetism. Then, starting from Faraday's law of induction and Maxwell's law of induction, derive the equation that describes a traveling plane electromagnetic wave. Here, you are asked to *explicitly* derive the wave equations for the electric field $E(x,t)$ and the magnetic field $B(x,t)$, and you are welcome to utilize the figure below that was discussed extensively in the class. (Note: Do not use the differential form of Maxwell's equations for this problem. You will have an opportunity to do so in other courses.)





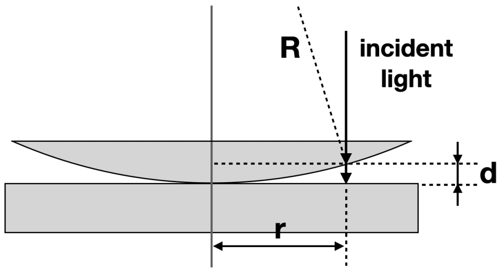


- (a) $\vec{\mu} = IA \cdot \hat{n} = \frac{e}{2\pi r/v} \cdot \pi r^2 \cdot \left(-\frac{\vec{L}}{|\vec{L}|}\right) = \frac{evr}{2} \cdot \frac{1}{mvr} \cdot (-\vec{L}) = -\frac{e}{2m}\vec{L}$. For details, see Chapter 32-5 of Halliday & Resnick.

- (b) $\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} \;\rightarrow\; hdE = -hdx\frac{dB}{dt} \;\rightarrow\; \frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$, while $\oint \vec{B} \cdot d\vec{s} = \mu_0\epsilon_0\frac{d\Phi_E}{dt} \;\rightarrow\; -hdB = \mu_0\epsilon_0\left(hdx\frac{dE}{dt}\right) \;\rightarrow\; -\frac{\partial B}{\partial x} = \mu_0\epsilon_0\frac{\partial E}{\partial t}$. Then, by combining the two resulting equations and noting $c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$, one can obtain $\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$ and $\frac{\partial^2 B}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2}$. For details, see Chapter 33-1 of Halliday & Resnick.

3. (a) [2 pt] A pencil with a height of 5.0 cm is placed 10 cm in front of a system of two thin lenses. Lens 1, which is closer to the pencil, has a focal length of $f_1 = -15\,\text{cm}$, lens 2 has $f_2 = 12\,\text{cm}$, and the lenses are separated by $d = 12\,\text{cm}$. For the image produced by lens 2, *(i)* how far is the image located from lens 2 (i.e., the image distance i_2 including its sign), and *(ii)* what is the height of the image? Also, *(iii)* what is the image type (real or virtual) and orientation (inverted with respect to the original object or not inverted)?

(b) [1 pt] When a planoconvex lens is placed on a flat glass plate, light incident from above creates circular interference fringes known as Newton's rings (see figure). This pattern is associated with the variable thickness of the air film between the lens and the glass plate. *(i)* For a lens with an index of refraction $n_1 = 1.5$ and radius of curvature $R = 6.0\,\text{m}$, and a glass plate with $n_2 = 1.8$, find the radius r of the third bright ring, in light of the wavelength $\lambda = 550\,\text{nm}$. *(ii)* If water with $n_3 = 1.3$ now fills the space between the lens and the plate, what is the new radius of this ring? (Note: You may assume $R \gg \lambda$. To keep our discussion simple, we do not consider the reflections from the top surface of the lens and the bottom of the glass plate.)



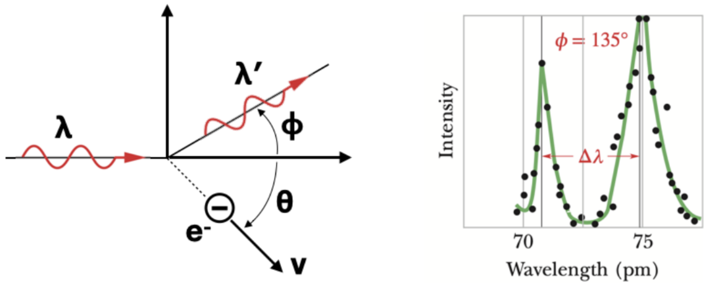
(c) [1 pt] In the class we discussed single-slit diffraction only for a slit, but a similar result holds when light bends around a straight, thin object such as a strand of hair or metal wire. Suppose that a He–Ne laser of wavelength $\lambda = 632.8\,\text{nm}$ illuminates a strand of hair, and a diffraction pattern appears on a screen at distance $L = 2.5\,\text{m}$. If the first dark fringes on either side of the central bright spot are $\Delta x = 4.8\,\text{cm}$ apart, how thick is this strand of hair? Describe how manufacturers of metal wire (or any other products with small dimensions) can use this idea to continuously monitor the thickness of their product. Use diagrams if desired.

- (a) $\frac{1}{f_1} = \frac{1}{p_1} + \frac{1}{i_1} \rightarrow \frac{1}{-15} = \frac{1}{10} + \frac{1}{i_1} \rightarrow i_1 = -6\,\text{cm}$. Then, with $p_2 = -i_1 + 12 = 18\,\text{cm}$, $\frac{1}{f_2} = \frac{1}{p_2} + \frac{1}{i_2} \rightarrow \frac{1}{12} = \frac{1}{18} + \frac{1}{i_2} \rightarrow i_2 = 36\,\text{cm}$, indicating that the image is real. From Eqs.(34-6) and (34-11) of Halliday & Resnick, the overall laternal magnification is $m = m_1 m_2 = \left(-\frac{i_1}{p_1}\right)\left(-\frac{i_2}{p_2}\right) = \left(-\frac{-6}{10}\right)\left(-\frac{36}{18}\right) = -1.2$, which means that the image is inverted with respect to the original.
- (b) For the bright Newton's rings, the path difference should be $2d = 2\left(R - \sqrt{R^2 - r^2}\right) = 2\left[R - R\left(1 - \frac{r^2}{R^2}\right)^{1/2}\right] \simeq \frac{r^2}{R} = \left(m + \frac{1}{2}\right)\frac{\lambda}{n_3}$, which is very much similar to Eq.(35-36) of Halliday & Resnick. Hence, for the third bright ring ($m = 2$), $r = \sqrt{\frac{5}{2} \cdot \frac{R\lambda}{n_3}}$, where n_3 is either 1.0 or 1.3.
- (c) From Eq.(36-3) of Halliday & Resnick, $a = \frac{m\lambda}{\sin\theta} \simeq \frac{m\lambda}{\tan\theta} \rightarrow a = \frac{\lambda}{\Delta x/2L}$.

4. (a) [1 pt] What are the kinetic energy K (in MeV), the total energy E (in MeV), and the momentum p (in MeV/c) for an electron moving at speed $v = 0.99c$?

(b) [1 pt] Near the top of the Earth's atmosphere, 120 km above sea level, a pion is created when an incoming high-energy cosmic ray particle collides with an atomic nucleus. The pion has a total energy $E = 1.4 \times 10^5\,\text{MeV}$ and descends vertically downward. If measured in a frame fixed with respect to the Earth, what altitude above sea level do you expect the pion to reach before it decays? (Note: The rest energy of a pion is 139.6 MeV. In a reference frame in which they are at rest, pions decay with an average lifetime of 26 ns. Ignore any general relativistic effects.)

(c) [2 pt] When a beam of X-rays of wavelength λ is directed onto a target of carbon, the scattered X-rays at an angle ϕ contain a range of wavelengths with two prominent intensity peaks at λ and λ' (see figure; $\lambda' > \lambda$). *(i)* Using the *relativistic* conservation of energy and momentum, derive the Compton shift formula, $\Delta\lambda = \lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\phi)$. *(ii)* Explain why we still observe the peak at the incident wavelength λ when ϕ is much larger than 0°.

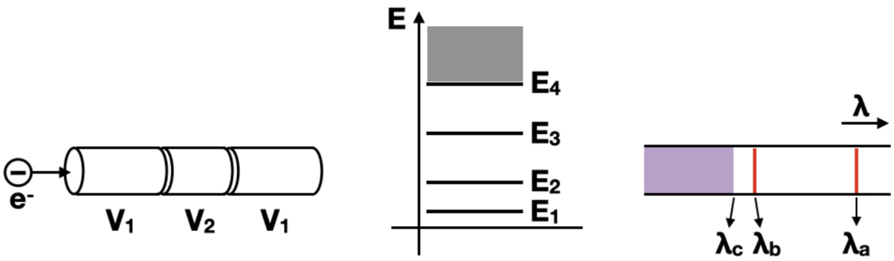


• (a) With $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $E = \gamma m_e c^2 = \gamma \times 0.511\,\mathrm{MeV}$, $K = (\gamma - 1) m_e c^2 = (\gamma - 1) \times 0.511\,\mathrm{MeV}$, and $p = \gamma m_e v = \gamma m_e c^2 (v/c^2) = \gamma m_e c^2 (\beta/c) = \gamma \times 0.511\,\mathrm{MeV/c} \times 0.99$.

• (b) From $\gamma = \frac{E}{m_\pi c^2}$, $\Delta t = \gamma \Delta t_0$ and $v = \beta c = c\sqrt{1 - \frac{1}{\gamma^2}} \simeq c$.

• (c) The relativistic energy and momentum conservation states that $\frac{h}{\lambda} = \frac{h}{\lambda'} + \left(\frac{E_e}{c} - m_e c\right)$, $\frac{h}{\lambda} = \frac{h}{\lambda'} \cos\phi + p_e \cos\theta$, and $0 = \frac{h}{\lambda'} \sin\phi - p_e \sin\theta$. The first equation gives $\left(\frac{E_e}{c}\right)^2 = \left(\frac{h}{\lambda} - \frac{h}{\lambda'} + m_e c\right)^2 = p_e^2 + m_e^2 c^2$, while the second and third equations are combined to yield $p_e^2 = \left(\frac{h}{\lambda} - \frac{h}{\lambda'} \cos\phi\right)^2 + \left(\frac{h}{\lambda'} \sin\phi\right)^2$. Rearranging, one can acquire $p_e^2 = \left(\frac{h}{\lambda} - \frac{h}{\lambda'} + m_e c\right)^2 - m_e^2 c^2 = \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)^2 + 2 m_e c \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)$, and $p_e^2 = \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)^2 + \frac{2h^2}{\lambda\lambda'} - \frac{2h^2}{\lambda\lambda'} \cos\phi$. Equating the last two equations leads to the desired formula, $\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\phi)$. For details — especially about the remaining peak at λ — see Chapter 38-3 of Halliday & Resnick.

5. (a) [1 pt] An electron travels to the right towards a thin tube shown in the figure below in which a one-dimensional finite potential well has been set up with voltages $V_1 < 0$ and $V_2 = 0$. The electron is then trapped in the well after losing energy, and is now in its ground state. The figure in the middle shows the energy-level diagram with the non-quantized region starting at $E_4 = 450\,\mathrm{eV}$. Shown on the right is the observed absorption spectrum of the ground-state electron, indicating the wavelengths of light it can absorb in transitions from this initial state via a single-photon absorption: $\lambda_a = 14.588\,\mathrm{nm}$ and $\lambda_b = 4.8437\,\mathrm{nm}$, and any wavelength less than $\lambda_c = 2.9108\,\mathrm{nm}$. Find the energy of the electron's first excited state.



(b) [2 pt] The energy states of an electron within an atom can be split due to the interaction of the electron's spin and the magnetic fields *inside* the atom. Here we discuss two examples. (Note: For both questions, you are welcome to make use of the convenient constant introduced in the textbook, the Bohr magneton $\mu_B = \frac{e\hbar}{2m_e} = 9.3 \times 10^{-24}\,\mathrm{J/T} = 5.8 \times 10^{-5}\,\mathrm{eV/T}$.)

(i) First, we consider the famous yellow light from an excited sodium atom as it transitions from $(n,l) = (3,1)$ to $(n,l) = (3,0)$. The yellow light is actually composed of two closely spaced spectral lines called the sodium doublet, $\lambda_1 = 589.0\,\mathrm{nm}$ and $\lambda_2 = 589.6\,\mathrm{nm}$. This is because the excited state at $(n,l) = (3,1)$ is in fact split into two levels depending on whether the electron's spin magnetic moment $\vec{\mu}_s$ is parallel or antiparallel to the internal magnetic field $\vec{B}_1$ associated with *the electron's orbital motion.* Find the effective strength of this internal magnetic field, B_1.

(ii) Now we consider a hydrogen atom in its ground state. Just like an electron, a proton (a hydrogen nucleus) is a charged particle and has a spin magnetic dipole moment, so it produces a magnetic field around itself, which we call $\vec{B}_2$. The electron's ground state at $(n,l) = (1,0)$ is then split into two levels depending on whether the electron's spin magnetic moment $\vec{\mu}_s$ is parallel or antiparallel to the internal magnetic field $\vec{B}_2$ associated with *the proton's spin.* When an electron transitions between these two closely spaced energy levels, it can absorb or emit light of $\lambda_3 = 21\,\mathrm{cm}$. Find the effective strength of this internal magnetic field, B_2.

- (a) $\frac{hc}{\lambda_a} = E_2 - E_1$, $\frac{hc}{\lambda_b} = E_3 - E_1$, $\frac{hc}{\lambda_c} = E_4 - E_1 \;\rightarrow\; E_2 = E_1 + \frac{hc}{\lambda_a} = E_4 - \frac{hc}{\lambda_c} + \frac{hc}{\lambda_a}$.
- (b-1) From Eqs.(32-24), (32-27), (40-13) and (40-22) of Halliday & Resnick, $\Delta E = \frac{hc}{\lambda_1} - \frac{hc}{\lambda_2} = 2\vec{\mu}_s \cdot \vec{B}_1 = 2\mu_{s,z} B_1 = 2\mu_B B_1$, which gives $B_1 \simeq 18\,\mathrm{T}$.
- (b-2) $\Delta E = \frac{hc}{\lambda_3} = 2\vec{\mu}_s \cdot \vec{B}_2 = 2\mu_{s,z} B_2 = 2\mu_B B_2$, which gives $B_2 \simeq 5.1 \times 10^{-2}\,\mathrm{T}$.

6. (a) [1 pt] Throughout the semester we discussed many examples in which concepts in classical physics are utilized in contemporary research or in explaining daily phenomena. In this regard, four of your peers presented their term projects at the end of the semester. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 3-4 sentences is expected to clearly convey the core physics idea of his/her term project. Use diagrams if desired. If you were one of the presenters, please choose someone else's.

(b) [1 pt] We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques such as order-of-magnitude estimation and/or dimensional analysis. Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem setup. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in $\sim$3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.

- (a) See the student presentation slides in Lecture 14-2 that include the collection of term project presentations by four students, and their video recordings on eTL.
- (b) See also the class slides for Lecture 13-2 that include many example problems, and the grading guideline.