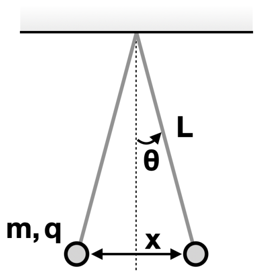
Physics II (Fall 2023): Midterm Exam Solution

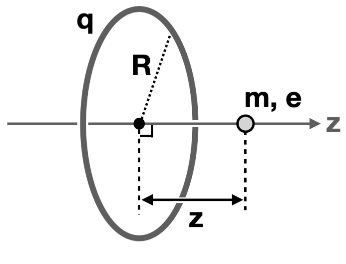
Oct. 20, 2023

[total 20 pts, closed book, 90 minutes]

1. (a) [1 pt] Two small conducting balls of equal mass $m = 6.0\,\mathrm{g}$ and equal charge q are suspended from massless, nonconducting cords of length $L = 120\,\mathrm{cm}$ (see figure). If the separation between the balls is $x = 4.0\,\mathrm{cm}$, determine $|q|$. You may assume that θ — the angle that the cord makes with the vertical — is so small that $\tan\theta$ can be approximated by $\sin\theta$. Then, explain what would happen if one of the balls were instantly discharged (e.g., if it lost q to the ground).



(b) [2 pt] An electron of mass m and charge e is constrained to move along the symmetry axis of a nonconducting ring of radius R on which positive charge q is uniformly distributed. Find the electric field (magnitude and direction) at the electron's position z (see figure) by *(i)* explicitly integrating the differential electric field, $\vec{E} = \int d\vec{E}$, and by *(ii)* first finding the electric potential and then differentiating V. Finally, show that when the electron is near the center of the ring ($z \ll R$), it oscillates through the center. Find the period of these small oscillations. (Note: You are asked to first derive the electron's equation of motion explicitly. But then, you may utilize the fact that the equation of motion in the form of $\ddot{z}(t) = -\omega^2 z(t)$ depicts an oscillatory motion of angular frequency ω.)

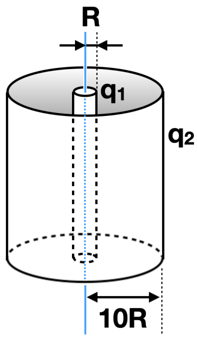


- (a) $mg\tan\theta \simeq mg \cdot \frac{x/2}{L} = \frac{1}{4\pi\epsilon_0}\frac{q^2}{x^2}$.

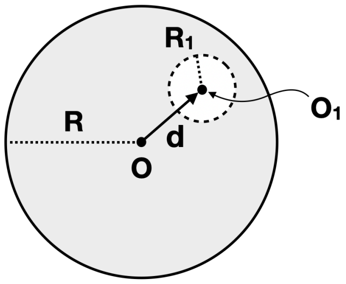
• (b-1) $E_z = \int dE \cos\theta = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{z\lambda R d\theta}{(z^2+R^2)^{3/2}} = \frac{1}{4\pi\epsilon_0}\cdot\frac{qz}{(z^2+R^2)^{3/2}}$ as in Eq.(22-16) of Halliday & Resnick. Meanwhile, $V = \int dV = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\lambda R d\theta}{(z^2+R^2)^{1/2}} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{(z^2+R^2)^{1/2}}$. Thus, $E_z = -\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0}\cdot\frac{qz}{(z^2+R^2)^{3/2}}$, which is equal to what we found earlier.

• (b-2) For $z \ll R$, the force on the electron becomes $F_z = m\ddot{z} = -eE \simeq -\frac{1}{4\pi\epsilon_0}\cdot\frac{eqz}{R^3}$ for $z > 0$, in which the minus sign implies that the force is towards the center. Thus, $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{4\pi\epsilon_0 m R^3}{eq}}$.

2. (a) [2 pt] A long conducting rod of radius $R = 1.2\,\mathrm{mm}$ and length $L = 20\,\mathrm{m}$ is inside a long, thin-walled coaxial conducting cylindrical shell of radius $10\,R$ (see figure below for a section of this coaxial system). The net charge on the rod is $q_1 = 3.2\times10^{-12}\,\mathrm{C}$, and that on the shell is $q_2 = -2\,q_1$. Determine the electric field $\vec{E}$ (magnitude and direction) at distance $r = 5\,R$ and $20\,R$ from the symmetry axis. What is the charge on the interior and exterior surface of the shell? (Note: With $L \gg R$, we neglect any fringing effect.)



(b) [1 pt] A sphere of uniform charge density ρ (negative charge, i.e., $\rho < 0$) and radius R has within it a spherical cavity of radius R_1 whose center is at $\vec{d}$ as shown in the figure below ($\vec{d}$ originates from the center of the sphere, O, and points to the center of the cavity, O_1). Using Gauss' law and the principle of superposition, show that the electric field inside the cavity is uniform and can be written as $\vec{E} = \frac{\rho}{3\epsilon_0}\,\vec{d}$.



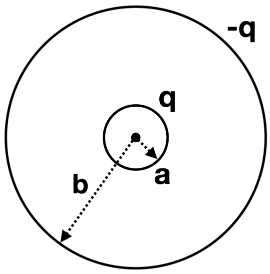
• (a) $\vec{E}_{(R<r<10R)} = \frac{q_{\rm enc}/L}{2\pi\epsilon_0 r}\hat{r} = \frac{q_1/L}{2\pi\epsilon_0 r}\hat{r}$ and $\vec{E}_{(r>10R)} = \frac{(q_1+q_2)/L}{2\pi\epsilon_0 r}\hat{r}$. Inside the conducting shell, $\vec{E}_{(10R-\epsilon<r<10R+\epsilon)} = 0 = \frac{(q_1+q_{2,\rm int})/L}{2\pi\epsilon_0 r}\hat{r}$, which therefore yields $q_{2,\rm int} = -q_1 = q_{2,\rm ext}$.

• (b) From Eq.(23-20) of Halliday & Resnick, the electric field $\vec{E}$ at an arbitrary point inside the cavity (with its position vector denoted as $\vec{r}$ originating from O, or as $\vec{r}_1$ originating from O_1) is $\vec{E}(\vec{r}) = \frac{\rho}{3\epsilon_0}\,\vec{r} + \frac{(-\rho)}{3\epsilon_0}\,\vec{r}_1 = \frac{\rho}{3\epsilon_0}\,(\vec{r}_1+\vec{d}) + \frac{(-\rho)}{3\epsilon_0}\,\vec{r}_1 = \frac{\rho}{3\epsilon_0}\,\vec{d}$.

3. (a) [1 pt] Positive charge q is uniformly distributed throughout the volume of a nonconducting sphere of radius R. Using Gauss' law, find the electric field as a function of distance r from the sphere's center, $\vec{E}(r)$, both inside and outside R. Then, by direct integration find the electric potential $V(r)$, both inside and outside R, while taking $V(\infty) = 0$. Sketch $V(r)$. What is the potential difference between the center of the sphere and the surface of the sphere?

(b) [2 pt] A metal sphere of radius a and charge q (> 0) is on an insulating stand at the center of a larger, spherical metal shell of radius b and charge $-q$ (see figure). Find the electric field as a function of distance r from the sphere's center, $\vec{E}(r)$, for all values of r ($0 < r < \infty$). Calculate and sketch $V(r)$, while taking $V(\infty) = 0$. What is the potential difference V_{ab} between the spheres? Finally, show that the capacitance of this two-sphere system is $C = 4\pi\epsilon_0 \left(\frac{ab}{b-a}\right)$.

[For an extra +0.5 point] Using your findings above, verify that the following equality holds for the energy stored in the electric field between the spheres: $\frac{1}{2}CV_{ab}^2 = \int_{\mathcal{V}} \frac{1}{2}\epsilon_0 \{E(r)\}^2 d\mathcal{V}$. (Note: In the class, we discussed this equality for the case of a parallel-plate capacitor. You may utilize $\int_{\mathcal{V}} f(r)d\mathcal{V} = \int_a^b f(r)4\pi r^2 dr$ for a spherically symmetric function $f(r)$ and volume $\mathcal{V}$.)



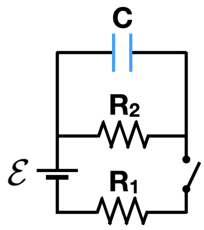
• (a) From Chapter 23-6 of Halliday & Resnick, $\vec{E}_{(r>R)} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$ and $\vec{E}_{(r<R)} = \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{r}$. Integrating, we get $V_{(r>R)} = -\int_\infty^r \frac{1}{4\pi\epsilon_0}\frac{qdr'}{r'^2} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$ and $V_{(r<R)} = -\int_\infty^R \frac{1}{4\pi\epsilon_0}\frac{qdr'}{r'^2} - \int_R^r \frac{1}{4\pi\epsilon_0}\frac{qr'dr'}{R^3} = \frac{1}{8\pi\epsilon_0}\frac{q}{R}\left(3 - \frac{r^2}{R^2}\right)$. Hence, we reach $V(0) - V(R) = \frac{1}{8\pi\epsilon_0}\frac{q}{R}$.

• (b-1) $\vec{E}_{(r<a)} = \vec{E}_{(r>b)} = 0$ while $\vec{E}_{(a<r<b)} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$. Integrating, $V_{(r>b)} = 0$, $V_{(a<r<b)} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{b}\right)$, and $V_{(r<a)} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$, which gives $V_{ab} = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$. Meanwhile, $C = 4\pi\epsilon_0\left(\frac{ab}{b-a}\right)$ can be found by following the steps that lead to Eq.(25-17) of Halliday & Resnick.

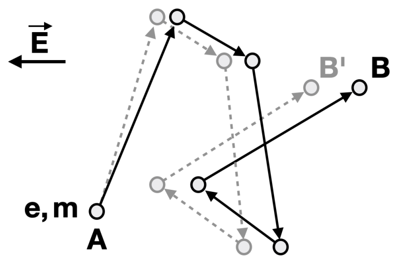
• (b-2) Thus, $\frac{1}{2}CV_{ab}^2 = \frac{1}{2}\cdot 4\pi\epsilon_0\left(\frac{ab}{b-a}\right)\cdot\left[\frac{q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)\right]^2 = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$, while $\int_{\mathcal{V}} \frac{1}{2}\epsilon_0 \{E(r)\}^2 d\mathcal{V} = \int_a^b \frac{1}{2}\epsilon_0\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$.

4. (a) [2 pt] In the circuit below, the resistances are $R_1 = 10\,\mathrm{k\Omega}$, $R_2 = 20\,\mathrm{k\Omega}$, the ideal battery has an emf of $\mathcal{E} = 30\,\mathrm{V}$, and the capacitor of $C = 0.40\,\mu\mathrm{F}$ is uncharged. *(i)* When the switch is closed, what is the current in R_1 and R_2 at that instant, respectively? *(ii)* A long time later (i.e., after a steady state has been reached), what is the current in R_2? *(iii)* Long after the steady state has been reached, the switch is opened at time t_0. What is the current in R_2 at

$t = t_0 + 4.0\,\text{ms}$? (Note: For *(iii)*, you are first asked to explicitly set up and solve the differential equation for the time variation of the charge q on the capacitor plates.)



(b) [2 pt] To understand why metals such as copper obey Ohm's law, we consider a model of the conduction process at the atomic level. Here we assume that the conduction electrons are free to move in the metal, and that they collide not with one another, but only with the metal atoms. The black lines in the figure below show a conduction electron of mass m and charge e moving from point A to B in the presence of an external electric field $\vec{E}$, making four collisions along the way. The gray lines from A to B' show what the electron's path would have been without the electric field. (Note: The drift velocity of the electron, v_d, shown in this figure is greatly exaggerated.) *(i)* Using this figure and variables such as the mean free time, τ, the number of electrons per unit volume, n, and the current density, $\vec{J}$, show that the resistivity of the metal is written as $\rho = \frac{m}{e^2 n\tau}$. *(ii)* Explain why this equation can be taken as a statement that metals obey Ohm's law. Also use the equation to explain the temperature dependence of the resistivity of a conductor.



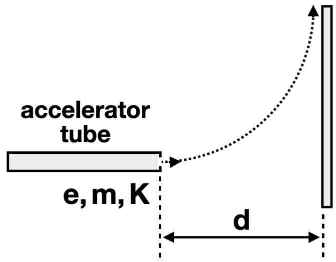
• (a-1) At the instant the switch is closed, $I_1 = 3.0 \times 10^{-3}$A and $I_2 = 0$. After a steady state has been reached, $I_2 = I_1 = \frac{\mathcal{E}}{R_1+R_2} = 1.0 \times 10^{-3}$A, which gives the charge stored on the capacitor $q_0 = CV_2 = C(I_2R_2)$.

• (a-2) $I(t) = \frac{dq(t)}{dt} = -\left(\frac{q_0}{R_2C}\right) e^{-(t-t_0)/R_2C}$ can be found by following the steps that lead to Eq.(27-40) of Halliday & Resnick, which gives $I(t_0 + 4.0\,\text{ms}) = 6.1 \times 10^{-4}$A.

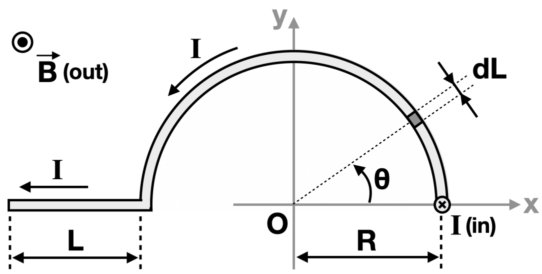
• (b) $v_d = \frac{J}{ne} = \frac{eE\tau}{m} \to \rho = \frac{E}{J} = \frac{m}{e^2n\tau}$. Since v_d is much smaller than the effective speed of the electron, τ is hardly affected by the field. Therefore, $\rho = \frac{m}{e^2n\tau}$ is independent of the field strength, proving that metals obey Ohm's law. The increase in resistivity with temperature is due to an increase in the collision rate, hence a decrease in τ. For details, see Chapters 26-4 and 26-5 of Halliday & Resnick.

5. (a) [1 pt] A beam of electrons — each with mass m, charge e, and kinetic energy K — emerges from a window at the end of an accelerator tube. A metal plate at distance d from

this window is perpendicular to the direction of the emerging beam (see figure). Treating the problem nonrelativistically, find the orientation and minimum strength of the uniform magnetic field $\vec{B}$ that must be applied in order to prevent the beam from hitting the plate.

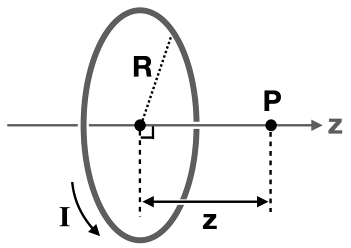


(b) [2 pt] The figure below depicts a conductor carrying current I to the left, immersed in a uniform magnetic field $\vec{B}$ that is pointing outward and perpendicular to this page. The conductor has three segments: *(i)* a straight segment of length L perpendicular to the plane of the page, *(ii)* a semi-circular segment of radius R, and *(iii)* another straight segment of length L parallel to the x-axis. What is the total magnetic force acting on this conductor (magnitude and direction)?

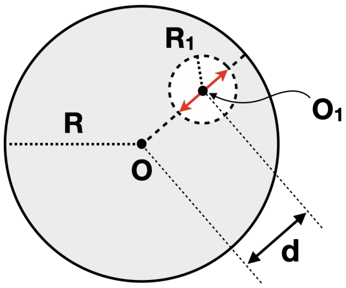


- (a) With the magnetic field B pointing out of the page, $r = \frac{mv}{eB} = \frac{\sqrt{2mK}}{eB} < d$, from Eq.(28-16) of Halliday & Resnick. Thus, the minimum strength of the magnetic field is $\frac{\sqrt{2mK}}{ed}$.
- (b) $\vec{F} = \int I d\vec{L} \times \vec{B} = \int_0^\pi I(Rd\theta)B\sin\theta\,\hat{y} + ILB\,\hat{y} = (2R + L)IB\,\hat{y}$.

6. (a) [2 pt] Consider a magnetic dipole, that is, a single circular current loop of radius R and current I. Find the the magnetic field (magnitude and direction) at point P on the symmetry axis of the loop, at distance z from the center of the loop (see figure), by explicitly integrating the differential magnetic field, $\vec{B} = \int d\vec{B}$. Explain which expressions your answer reduces to *(i)* when P is at the center of the loop, and *(ii)* when P is very far from the loop ($z \gg R$). (Note: For *(ii)*, you are asked to define *and* include the magnetic dipole moment $\vec{\mu}$ in your expression.)



(b) [2 pt] The figure below shows a cross-section of a long cylindrical conductor of radius R containing a long cylindrical hole of radius R_1. The central axes of the cylinder and the hole are parallel and are distance d apart, and current I is uniformly distributed over the tinted, conducting area.



Now consider a straight line $\overleftrightarrow{OO_1}$ in this cross-section, connecting the central axis of the cylinder, O, and the central axis of the hole, O_1, and extending beyond both points. *(i)* Using Ampere's law and the principle of superposition, show that the magnetic field along $\overleftrightarrow{OO_1}$ inside the hole (i.e., the red line segment in the figure) is uniform with a strength $|\vec{B}| = \frac{\mu_0 I d}{2\pi(R^2 - R_1^2)}$. *(ii)* Do a sanity check by discussing the two special cases: $R_1 = 0$, and $d = 0$.

• (a) $B_z = \int dB \sin\theta = \frac{\mu_0}{4\pi} \int_0^{2\pi} \frac{IR \cdot Rd\theta}{(z^2+R^2)^{3/2}} = \frac{\mu_0 I R^2}{2(z^2+R^2)^{3/2}}$ as in Eq.(29-26) of Halliday & Resnick. When $z = 0$, the equation reduces to Eq.(29-9) of Halliday & Resnick with $\phi = 2\pi$; when $z \gg R$, the equation becomes Eq.(29-27) of Halliday & Resnick.

• (b) From Eq.(29-20) of Halliday & Resnick, the magnetic field strength $|\vec{B}|$ at point P on the red line segment at distance r from O is

$$|\vec{B}(r)| = \begin{cases} \dfrac{\mu_0 I \left(\frac{R^2}{R^2 - R_1^2}\right)}{2\pi R^2} \cdot r + \dfrac{\mu_0 I \left(\frac{R_1^2}{R^2 - R_1^2}\right)}{2\pi R_1^2} \cdot (d - r), & \text{if } r < d, \\[2ex] \dfrac{\mu_0 I \left(\frac{R^2}{R^2 - R_1^2}\right)}{2\pi R^2} \cdot r - \dfrac{\mu_0 I \left(\frac{R_1^2}{R^2 - R_1^2}\right)}{2\pi R_1^2} \cdot (r - d), & \text{if } r > d, \end{cases}$$

which yields $|\vec{B}| = \frac{\mu_0 I d}{2\pi(R^2 - R_1^2)}$ regardless. If $R_1 = 0$, then the equation simply reduces to Eq.(29-20) of Halliday & Resnick. If $d = 0$, then it becomes $B = 0$ inside the hole, obviously.