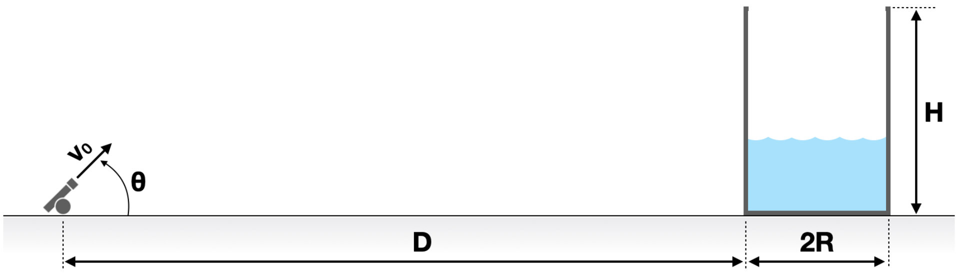
Physics I (Spring 2023): Midterm Exam Solution

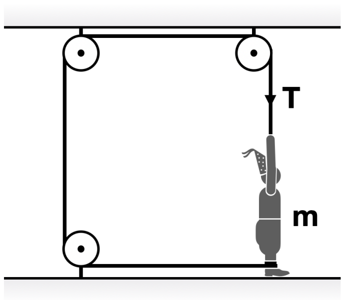
Apr. 21, 2023

[total 25 pts, closed book, 90 minutes]

1. (a) [2 pt] A firefighter uses a water cannon to fill a cylindrical storage tank of radius $R = 2.0$ m and height $H = 6.0$ m, with its top fully open. The cannon is located $D = 18.0$ m away from the tank (see figure), and the water is shot at angle $\theta = 45°$ above the horizontal from the same level as the base of the tank. For what *range* of launch speeds v_0 will the water enter the tank?

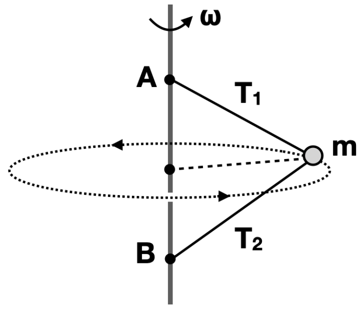


(b) [2 pt] A professional clown attempts to get laughs with classic slapstick comedy. He is about to pull a rope vertically downward that passes around three pulleys and is then tied around his ankle (see figure). The clown weighs $m = 75$ kg, and the coefficient of static friction between the clown's feet and the ground is $\mu_s = 0.52$. Each pulley has negligible mass and negligible friction on its axle. What is the minimum force magnitude T that the clown must pull with in order to yank his feet out from under himself?

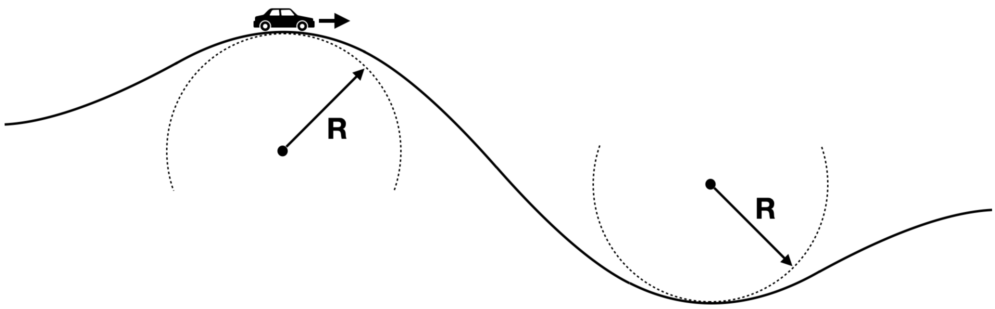


- (a) The vertical displacement is $(v_{0,\text{min}} \sin\theta)t_1 - \frac{1}{2}gt_1^2 = H$ when $t_1 = \frac{D}{v_{0,\text{min}} \cos\theta}$, whereas $(v_{0,\text{max}} \sin\theta)t_2 - \frac{1}{2}gt_2^2 = H$ when $t_2 = \frac{D+2R}{v_{0,\text{max}} \cos\theta}$.
- (b) $T = \mu_s F_N = \mu_s(mg - T)$.

2. (a) [2 pt] A steel ball of mass $m = 3.0$ kg and negligible size is attached to two points (A and B) of a massless vertical rod by means of two massless strings of the same lengths $l = 2.5$ m. Point A and B are 3.0 m apart (see figure), and the whole system rotates about the axis of the rod at a constant angular speed while both strings remain tight. If the tension T_1 in the upper string is measured to be 80 N, what is the tension T_2 in the lower string? How many revolutions per minute does the system make?



(b) [2 pt] A stunt car is driven at constant speed over a hill and then into a valley (see figure). The crest of the hill and the trough of the valley are both approximated by a circle of the same radius $R = 20$ m. What is the maximum speed that the car could have at the top of the hill to remain in contact with it there? What is the normal force on the driver ($m = 60$ kg) from the car seat there? Assuming that the car is driven at this speed since then, what is the normal force on the driver from the seat when the car passes through the bottom of the valley? To realize the motion discussed here, what does the driver need to do as the car descends from the top of the hill into the valley (i.e., can he simply do nothing with his foot off the brake or the gas pedal)?

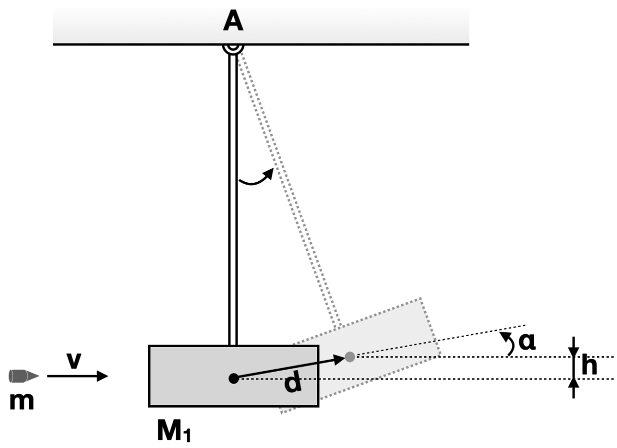


- (a) $T_1 \sin\theta = mg + T_2 \sin\theta$ where $\sin\theta = \frac{1.5}{2.5} = 0.6$, and $(T_1 + T_2)\cos\theta = m(l\cos\theta)\,\omega^2$.
- (b) $F_{N,\text{crest}} - mg = -\frac{mv^2}{R}$ at the top of the hill, for which $F_{N,\text{crest}} = 0$ yields $v = v_{\text{max}} = \sqrt{gR}$. Putting $v = \sqrt{gR}$ in $F_{N,\text{trough}} - mg = \frac{mv^2}{R}$ at the bottom of the valley gives $F_{N,\text{trough}} = 2mg$.

3. A ballistic pendulum consists of a block of wood of mass $M_1 = 4.0$ kg hanging from a hinged rod that can swing without friction on the horizontal axle (point A).

(a) [1 pt] Let us for the time being assume that the hinged rod is massless. A bullet of mass $m = 15$ g is fired into the block with an unknown constant speed v, coming quickly to rest inside the block. The collision is so brief that we can assume that the *block+bullet* system is isolated during the collision, and the collision is 1-dimensional. Immediately after the collision, the block

— with the bullet at its center — begins to swing upward (see figure). Their center of mass (CoM) rises a vertical distance $h = 6.0$ cm before the pendulum comes to rest momentarily. Determine v using the numerical values given herein.



(b) [1 pt] During the block's ascent to h right after the collision, *(i)* how much work is done on the *block+bullet* system by the gravitational force? (Note: You are asked to explicitly evaluate the amount of work using $W = \vec{F} \cdot \vec{d}$ where $\vec{d}$ is the displacement of the CoM noted in the figure. Here we assumed that α and h are so small that the arc along which the CoM moves can be approximated as a straight vector $\vec{d}$.) Using your result, *(ii)* show that the negative of the work done on the block by the gravitational force is the change in its gravitational potential energy, ΔU.

(c) [1 pt] During the block's ascent to h right after the collision, show that the net work done on the block is the change in its kinetic energy, ΔK. Why do you not need to consider the work done on the *block+bullet* system by the force of the rod?

(d) [2 pt] Now let us abandon the assumption that the hinged rod is massless, and re-estimate the bullet's speed v. Assume that the uniform thin rod has mass $M_2 = 1.0$ kg and length $l = 60$ cm, while the block M_1 can be treated as a particle (i.e., the size of the block is small compared to l). *(i)* Then, what is the rotational inertia of the *block+bullet+rod* system about point A? *(ii)* If the angular speed of the *block+bullet+rod* system about point A immediately after the collision is measured to be 1.6 rad/s, determine v using the numerical values given herein. (Note: The table of various rotational inertias is in the last page of this exam.)

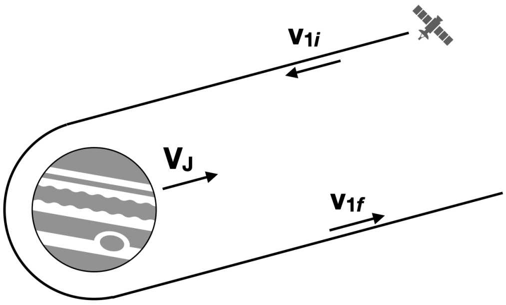
• (a) $(m + M_1)gh = \frac{1}{2}(m + M_1)V^2 = \frac{1}{2}\frac{m^2}{m+M_1}v^2 \quad \rightarrow \quad v = \sqrt{2gh} \cdot \frac{m+M_1}{m} \simeq 290$ m/s.

• (b) $W_g = (m + M_1)\vec{g} \cdot \vec{d} = (m + M_1)gd\cos(90^\circ + \alpha) = -(m + M_1)gd\sin\alpha = -(m + M_1)gh$. Meanwhile, $W_r = 0$ since the force from the rod is always perpendicular to the block's velocity.

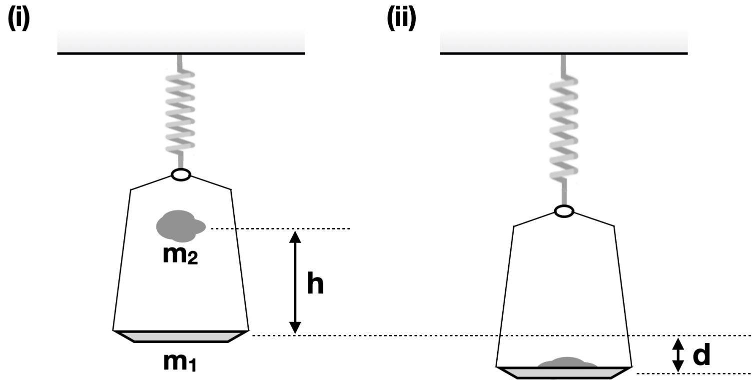
• (c) The force exerted by the rod on the block is always directed perpendicular to the block's direction of travel.

• (d) $I = (m+M_1)l^2 + \left\{\frac{1}{12}M_2 l^2 + M_2(\frac{l}{2})^2\right\} \simeq 1.6\ \mathrm{kg\,m^2}$. From angular momentum conservation we find $v = \frac{I\omega}{ml} \simeq 280$ m/s.

4. (a) [2 pt] A spacecraft of mass $m_1 = 9{,}500$ kg and speed $v_{1i} = 9.9$ km/s approaches Jupiter of mass $m_2 = 1.9 \times 10^{27}$ kg and speed $V_J = 13$ km/s. The two velocities are relative to the Sun, and are in opposite directions (see figure). The spacecraft then rounds the planet, and departs with speed v_{1f} (relative to the Sun) in the opposite direction. What is the magnitude of v_{1f} after this slingshot encounter which can be approximated as an elastic collision?



(b) [2 pt] When suspended from a coil spring of negligible mass, a steel plate of mass $m_1 = 150\,\text{g}$ (with frames of negligible mass) stretches the spring by $d_0 = 15\,\text{cm}$. Then, a lump of sticky clay of mass $m_2 = 200\,\text{g}$ is dropped from rest onto the plate from a height of $h = 30\,\text{cm}$ (see figure). Find the maximum distance d the plate travels downward from its initial equilibrium position.

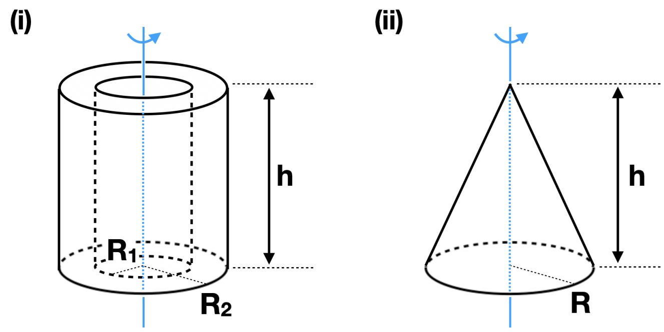


• (a) From Eqs.(9-75)-(9-76) of Halliday & Resnick with $m_2 \gg m_1$ and $v_{2i} = V_J$, we find $v_{2f} = V_J$ (obviously) and $v_{1f} = -v_{1i} + 2V_J = -9.9 + 2(-13) \simeq -36$ km/s.

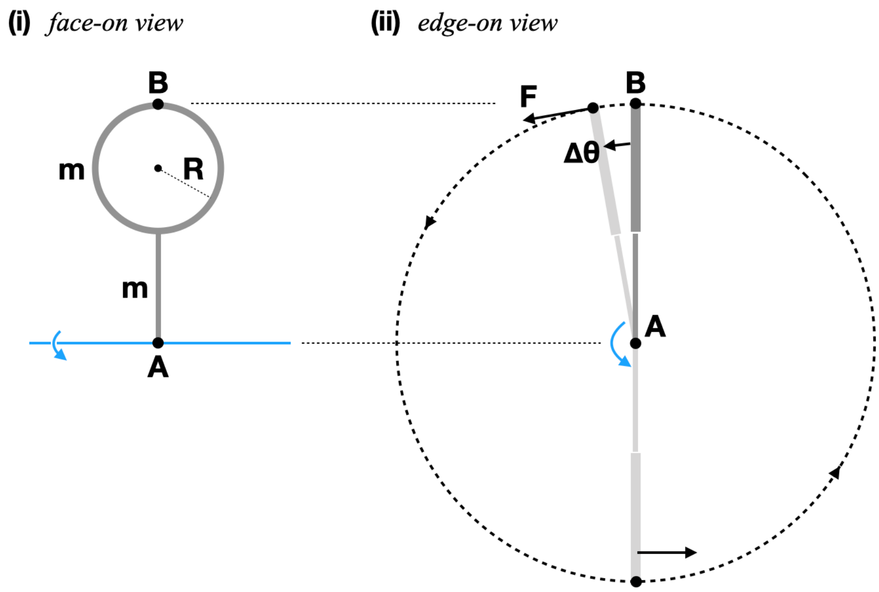
• (b) $\frac{1}{2}k(d+d_0)^2 = \frac{1}{2}kd_0^2 + (m_1+m_2)gd + \frac{1}{2}(m_1+m_2)V^2 = \frac{1}{2}kd_0^2 + (m_1+m_2)gd + \frac{1}{2}\frac{m_2^2}{m_1+m_2}v^2$, where $v = \sqrt{2gh}$ is the speed of m_2 right before it collides into m_1, V is the speed of $m_1 + m_2$ system immediately after the collision, and $k = \frac{m_1 g}{d_0}$ is the spring constant. By plugging the numerical values given, one may find the quadratic equation $4.9d^2 - 1.96d - 0.336 = 0$ (each term in units of J), which yields $d \simeq 0.53$ m.

5. (a) [2 pt] Using integration, calculate the moment of inertia of each of the following objects about its axis of symmetry (see figure): *(i)* an annular cylinder of mass M and uniform density with height h, inner radius R_1, and outer radius R_2, *(ii)* a solid cone of mass M and uniform density with height h, and the radius of its circular base R. (Note: For *(i)*, you may already

find the answer in the table of rotational inertias in the last page of this exam. For *(ii)*, you may want to utilize the rotational inertia of a uniform disk listed in the same table.)



(b) [2 pt] A rigid assembly of a thin hoop (of mass $m = 1.0$ kg and radius $R = 15$ cm) and a thin rod (of mass m and length $2R$) is positioned upright (see figure). It can rotate around a horizontal, frictionless axle which is in the plane of the hoop, through the lower end of the rod (point A). Now, in order to initiate a rotation, a constant force $\vec{F} = 49$ N $(= 5\,mg)$ is exerted on the top end of the hoop (point B) perpendicular to the hoop's plane at all times, while the assembly rotates by $\Delta\theta = 0.1$ rad. What are the magnitudes of the assembly's angular velocity $\vec{\omega}$ and angular momentum $\vec{L}$ about the rotation axis when it passes through the inverted (upside-down) orientation? (Note: The table of various rotational inertias is in the last page of this exam.)



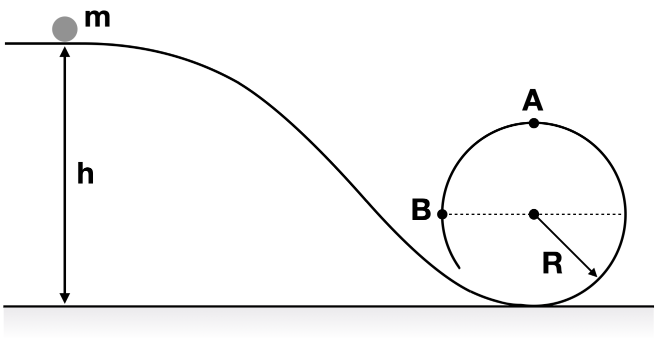
- (a-1) $I_1 = \displaystyle\int r^2 dm = \int_{R_1}^{R_2} r^2 \cdot \rho\,(2\pi r dr)h, \quad \text{where } \rho = \frac{M}{\pi(R_2^2 - R_1^2)h} \quad \rightarrow \quad I_1 = \frac{1}{2}M(R_1^2 + R_2^2).$

- (a-2) $I_2 = \displaystyle\int dI_{\text{disk}} = \int_0^h \frac{1}{2}\left(\frac{zR}{h}\right)^2 \cdot \rho\pi\left(\frac{zR}{h}\right)^2 dz, \quad \text{where } \rho = \frac{M}{(1/3)\pi R^2 h} \quad \rightarrow \quad I_2 = \frac{3}{10}MR^2.$

- (b) $W = 2mg\Delta z + \displaystyle\int \tau\Delta\theta = 2mg \cdot 4R + 5mg \cdot 4R \cdot \Delta\theta = 10mgR$

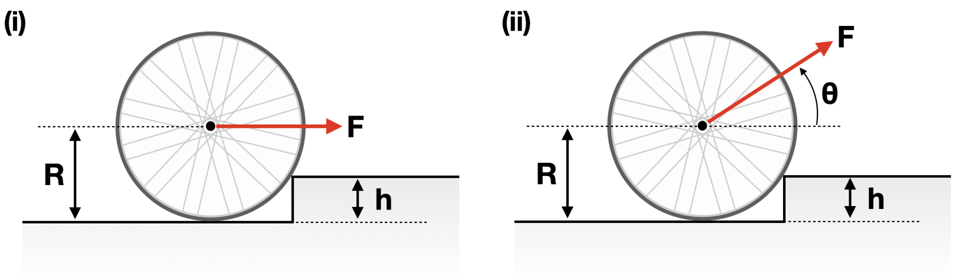
$$= \frac{1}{2}\left[\left\{\frac{1}{12}m(2R)^2 + mR^2\right\} + \left\{\frac{1}{2}mR^2 + m(3R)^2\right\}\right]\omega^2 \quad \rightarrow \quad \omega = \sqrt{\frac{24g}{13R}} \simeq 11\,\text{rad/s}.$$

6. (a) [2 pt] A thin-walled, hollow spherical shell of mass m starts from rest and rolls smoothly without slipping down a track (see figure). The diameter of the shell, $2r$, is very small compared to the radius of the circular loop, R. The work done by rolling friction is negligible. *(i)* What is the shell's initial height h (in units of R) for which this shell will be on the verge of leaving the track when reaching the top of the loop (point A)? *(ii)* In this case, how hard does the track push on the shell (in units of mg) at point A? How about at point B, which is at the same level as the center of the circle? What is the direction of this force acting on the shell at each point?



Now, suppose that the track had no friction and the shell was released from the same height h you found above. *(iii)* Would it make a complete loop-the-loop on the circular part of the track? How do you know without even explicitly calculating the numerical values? *(iv)* In this case, how hard would the track push on the shell (in units of mg) at point A and at point B, respectively? (Note: The table of various rotational inertias is in the last page of this exam.)

(b) [2 pt] A titanium wheel of mass $m = 5.0$ kg and radius $R = 20$ cm is resting against a step of height $h = 10$ cm. *(i)* A horizontal force $\vec{F}$ is applied at the axle of the wheel to roll it over the step obstacle (see figure). What is the minimum force magnitude necessary? (Note: As the magnitude of $\vec{F}$ increases, there comes a moment when the wheel begins to rise up and loses contact with the ground.) *(ii)* Now, in a different scenario, a constant force $\vec{F}$ is applied at the axle of the wheel at angle θ above the horizontal, just enough to start lifting the wheel off the ground (see figure). What is the value of θ that requires the least magnitude of the force $\vec{F}$? At this angle θ you just found, what is the magnitude of $\vec{F}$?



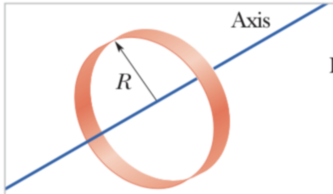
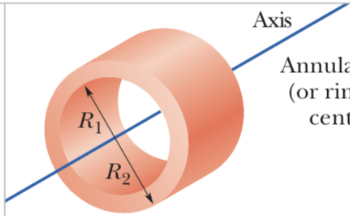
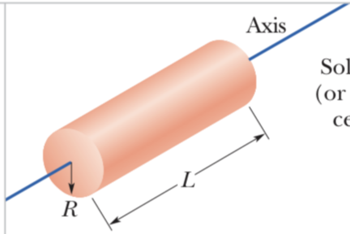
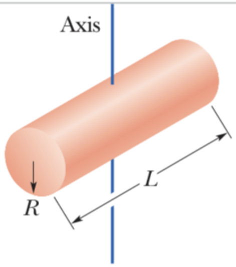
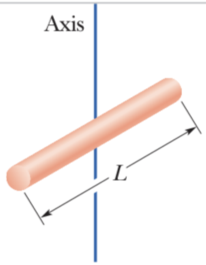
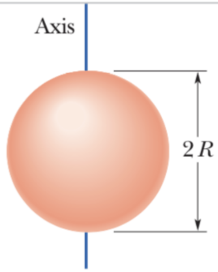
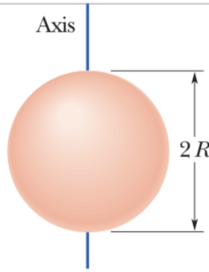
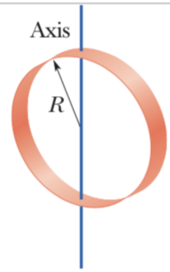
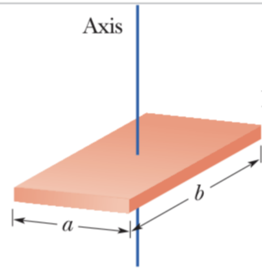
• (a-1) At point A, the centripetal force is $F_{\text{cen},A} = F_{N,A} - mg = -\frac{mv_A^2}{R}$, from which one can find that, on the verge of losing contact with the track ($F_{N,A} = 0$), the shell's speed must be $v_A = \sqrt{gR}$. Then, $mgh = mg(2R) + \frac{1}{2}mv_A^2 + \frac{1}{2}\left(\frac{2}{3}mr^2\right)\left(\frac{v_A}{r}\right)^2 = mg(2R) + \frac{5}{6}mv_A^2 = \frac{17}{6}mgR$ yields $h = \frac{17}{6}R$. At point B, $mg\left(\frac{17}{6}R\right) = mgR + \frac{5}{6}mv_B^2$ gives $F_{\text{cen},B} = F_{N,B} = -\frac{mv_B^2}{R} = -\frac{11}{5}mg$ where the negative sign indicates a direction towards the center of the circle.

• (a-2) Without friction on the track, the shell's kinetic energy is now $\frac{1}{2}mv^2$ instead of $\frac{5}{6}mv^2$, which makes it to move faster when released from the same height. At point A, $mg\left(\frac{17}{6}R\right) = mg(2R) + \frac{1}{2}mv_A^2$ gives $F_{\text{cen},A} = F_{N,A} - mg = -\frac{mv_A^2}{R} = -\frac{5}{3}mg$, and thus, $F_{N,A} = -\frac{2}{3}mg$. At point B, $mg\left(\frac{17}{6}R\right) = mgR + \frac{1}{2}mv_B^2$ gives $F_{\text{cen},B} = F_{N,B} = -\frac{mv_B^2}{R} = -\frac{11}{3}mg$.

• (b-1) The requirements for equilibrium are $F_{E,x}+F=0$, $F_{E,y}-mg=0$, and $-FR\sin 150°+mgR\sin 120° = 0$, where F_E is the force from the step edge towards the wheel's center. Either the third equation or the first and second equations combined gives $F = \frac{mg\cos 30°}{\sin 30°} \simeq 85$ N.

• (b-2) The requirements for equilibrium are $F_{E,x} + F\cos\theta = 0$, $F_{E,y} + F\sin\theta - mg = 0$, and $-FR\sin(150° - \theta) + mgR\sin 120° = 0$. The third equation transforms to $F = \frac{mg\sin 120°}{\sin(150°-\theta)} = \frac{mg\cos 30°}{\sin(30°+\theta)} \geq mg\cos 30°$, where the equality is when $\theta = 60°$. Combined, the first and second equations can also yield the same inequality by way of $\frac{F_{E,y}}{F_{E,x}} = \frac{mg-F\sin\theta}{-F\cos\theta} = -\tan 30°$. At $\theta = 60°$, the minimum force magnitude needed to roll the wheel over the step is $F = mg\cos 30° \simeq 42$ N.

Table 10-2 Some Rotational Inertias

 Hoop about central axis $I = MR^2$ (*a*)	 Annular cylinder (or ring) about central axis $I = \frac{1}{2}M(R_1^2 + R_2^2)$ (*b*)	 Solid cylinder (or disk) about central axis $I = \frac{1}{2}MR^2$ (*c*)
 Solid cylinder (or disk) about central diameter $I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$ (*d*)	 Thin rod about axis through center perpendicular to length $I = \frac{1}{12}ML^2$ (*e*)	 Solid sphere about any diameter $I = \frac{2}{5}MR^2$ (*f*)
 Thin spherical shell about any diameter $I = \frac{2}{3}MR^2$ (*g*)	 Hoop about any diameter $I = \frac{1}{2}MR^2$ (*h*)	 Slab about perpendicular axis through center $I = \frac{1}{12}M(a^2 + b^2)$ (*i*)