Mathematical Physics I (Fall 2025): Homework #3 Solution

Due Oct. 17, 2025 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

1. Boas Chapter 4, Problem 7.28

(Note: For Problem 7.28, you may want to prove and utilize the findings in Problem 7.27. Note the meaning of the subscripts next to the partial derivatives from Boas Chapter 4, Section 1 in case you did not read it. Prove further that the resulting formula is the familiar $c_p - c_v = nR$ found in e.g., Schroeder or Halliday & Resnick, where n is the number of moles of gas present and R is the gas constant.)

• Replacing $x \to T$, $y \to v$, $z \to p$ and $u \to s$ gives $\left(\frac{\partial s}{\partial T}\right)_p = \left(\frac{\partial s}{\partial T}\right)_v + \left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_p$, thus $c_p = c_v + T\left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_p$. With $\left(\frac{T\partial s}{\partial v}\right)_T = p$ and $\left(\frac{\partial v}{\partial T}\right)_p = \left(\frac{\partial(nRT/p)}{\partial T}\right)_p = \frac{nR}{p}$, you find $c_p - c_v = nR$.

2. Boas Chapter 4, Problem 9.5

• From Thm.(9.14) of Boas Chapter 4, we write $F = 4x^2 + y^2 + z^2 + \lambda(2x + 3y + z)$. Then, $\frac{\partial F}{\partial x} = 8x + 2\lambda = 0$, $\frac{\partial F}{\partial y} = 2y + 3\lambda$, and $\frac{\partial F}{\partial z} = 2z + \lambda = 0$. Combining the equations with the constraints $2x + 3y + z = 11$ gives $\lambda = -2$ and $(x, y, z) = (\frac{1}{2}, 3, 1)$.

3. Boas Chapter 4, Problem 12.15

(Note: For Problem 12.15, you may want to carefully review Boas Chapter 4, Section 12. You are also asked to prove the first equality, $\int_0^\infty e^{-ax}\sin kx\,dx = \frac{k}{a^2+k^2}$. The relations you prove here will be particularly useful later in Boas Chapter 8, Section 8, as we discuss the *Laplace transform.*)

• The first equality can be readily derived either through integration by parts or by the method demonstrated in Example 3 of Boas Chapter 8, Section 8. The second and third equations can be proven using Eq.(12.9) from Chapter 4, Section 12, as illustrated in Example 4 of Chapter 8, Section 8.

4. Boas Chapter 5, Problem 4.16

(Note: For Problem 4.16, also find $\partial(u,v)/\partial(x,y)$, for which you may want to prove and utilize the first theorem in Problem 4.18.)

• From Eq.(4.8) with $x = \frac{1}{2}(u^2 - v^2)$ and $y = uv$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} u & -v \\ v & u \end{vmatrix} = u^2 + v^2,$$

whereas, from the findings in Problem 4.18, the other Jacobian of variables u, v with respect to x, y is

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right)^{-1} = \frac{1}{u^2 + v^2} = \frac{1}{2(x^2 + y^2)^{1/2}}.$$

5. Boas Chapter 5, Problem 6.27

(Note: For Problem 6.27, you will first have to show that the intervals of integration for u and v are $[0, 1]$ and $[0, 1 + u]$, respectively. See the hint in Problem 4.20 for more information.)

• From Eq.(4.8) with $x = \frac{v}{1+u}$ and $y = \frac{uv}{1+u}$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{-v}{(1+u)^2} & \frac{1}{1+u} \\ \frac{v}{(1+u)^2} & \frac{u}{1+u} \end{vmatrix} = -\frac{v}{(1+u)^2},$$

which makes the integral

$$\int_0^1 dx \int_0^x \frac{(x+y)e^{x+y}}{x^2} dy = \int_0^1 du \int_0^{1+u} \frac{ve^v}{(v/1+u)^2} |J| dv = \int_0^1 du \int_0^{1+u} e^v dv$$
$$= [e^{1+u} - u] \Big|_0^1 = e^2 - e - 1.$$

6. Boas Chapter 6, Problem 3.13

(Note: For Problem 3.13, tackle the problem in two different ways, first by writing in ordinary vector notation, then by writing in tensor notation as described in Boas Chapter 10, Section 5. For the tensor approach, you may utilize the result from Problems 5.10 and 5.11 of Boas Chapter 10.)

• Letting $\mathbf{B} \times \mathbf{C} = \mathbf{D}$ and $\mathbf{C} \times \mathbf{A} = \mathbf{E}$, Thm.(3.3) and (3.8) together give $[(\mathbf{A} \times \mathbf{B}) \times \mathbf{D}] \cdot \mathbf{E} = -[\mathbf{D} \times (\mathbf{A} \times \mathbf{B})] \cdot \mathbf{E} = -[\cancel{(\mathbf{D} \cdot \mathbf{B})}\mathbf{A} - (\mathbf{D} \cdot \mathbf{A})\mathbf{B}] \cdot \mathbf{E} = (\mathbf{D} \cdot \mathbf{A})(\mathbf{B} \cdot \mathbf{E}) = [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})][\mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})] = [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})]^2 = (\mathbf{ABC})^2$.

• Alternatively, again with $\mathbf{B} \times \mathbf{C} = \mathbf{D}$ and $\mathbf{C} \times \mathbf{A} = \mathbf{E}$, writing in tensor notation and then using Eqs. (5.8), (5.11) and (5.12) of Boas Chapter 10, $\{[(\mathbf{A} \times \mathbf{B}) \times \mathbf{D}] \cdot \mathbf{E}\}_n = [(\mathbf{A} \times \mathbf{B}) \times \mathbf{D}]_n E_n = \epsilon_{nip}(\mathbf{A} \times \mathbf{B})_i D_p E_n = \epsilon_{nip}(\epsilon_{ijk} A_j B_k) D_p E_n = \epsilon_{ipn}\epsilon_{ijk} A_j B_k D_p E_n = (\delta_{pj}\delta_{nk} - \delta_{pk}\delta_{nj}) A_j B_k D_p E_n = (A_j D_j)(B_k E_k) - (A_j E_j)(B_k D_k) = (A_j \epsilon_{jlm} B_l C_m)(B_k \epsilon_{klm} C_l A_m) - \cancel{(A_j \epsilon_{jlm} C_l A_m)}\cancel{(B_k \epsilon_{klm} B_l C_m)} = (\epsilon_{klm} A_k B_l C_m)^2 = [\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})]^2 = (\mathbf{ABC})^2$

7. Boas Chapter 6, Problem 6.16

• Using Eq.(9.6) of Chapter 4 and Eq.(6.2) of Chapter 6, we write $F = \frac{d\phi}{ds} + \lambda(a^2 + b^2 + c^2) = \left(\frac{\partial\phi}{\partial x}a + \frac{\partial\phi}{\partial y}b + \frac{\partial\phi}{\partial z}c\right) + \lambda(a^2 + b^2 + c^2)$. Then, $\frac{\partial F}{\partial a} = \frac{\partial\phi}{\partial x} - 2\lambda a = 0$, $\frac{\partial F}{\partial b} = \frac{\partial\phi}{\partial y} - 2\lambda b = 0$, $\frac{\partial F}{\partial c} = \frac{\partial\phi}{\partial z} - 2\lambda c = 0$, and $a^2 + b^2 + c^2 = 1$. Combined, $\lambda = \pm\frac{|\nabla\phi|}{2}$ and $\mathbf{u} = (a, b, c) = \frac{\nabla\phi}{2\lambda} = \pm\frac{\nabla\phi}{|\nabla\phi|}$. With Eq.(6.4), you get the extremum values of $\frac{d\phi}{ds}$ as $\nabla\phi\cdot\mathbf{u} = \pm\frac{(\nabla\phi)^2}{|\nabla\phi|} = \pm|\nabla\phi|$.

8. Boas Chapter 6, Problem 9.7

(Note: For Problem 9.7, you are asked to prove and utilize the findings in Problem 9.6.)

• The area inside the elliptical curve can be written with the parametrization $x = a\cos\theta$ and $y = b\sin\theta$ as

$$\begin{aligned}\iint dxdy &= \frac{1}{2}\oint(xdy - ydx) = \frac{1}{2}\oint[a\cos\theta\cdot b\cos\theta d\theta - b\sin\theta\cdot(-a\sin\theta)d\theta]\\ &= \frac{1}{2}\int_0^{2\pi} ab(\sin^2\theta + \cos^2\theta)\,d\theta = \pi ab.\end{aligned}$$

9. (a) Review Boas Chapter 6, Example 7.2, thus prove Eq.(7.6) or Eq.(f) in p.339. You are also asked to prove the identity by explicitly working with the components, i.e.,

$$\nabla\cdot(\phi\mathbf{V}) = \frac{\partial(\phi V_x)}{\partial x} + \frac{\partial(\phi V_y)}{\partial y} + \frac{\partial(\phi V_z)}{\partial z} = ...$$

(b) Review how Gauss's law $\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$ is acquired from Coulomb's law in Boas Chapter 6, Section 10, where $\mathbf{E}$ is the electric field, ρ is the volume charge density, and ϵ_0 is the permittivity of free space.

(c) Using (a) and (b), show that the energy of a continuous charge distribution is given by

$$W_\mathrm{e} = \frac{1}{2}\int\rho V d\tau = \frac{\epsilon_0}{2}\int|\mathbf{E}|^2 d\tau$$

for integration over all space, where V is the electric potential. You are first asked to briefly discuss how one arrived at the term $W_\mathrm{e} = \frac{1}{2}\int\rho V d\tau$, and then explicitly prove the second equality. You may assume that V vanishes at large distance r at least as fast as r^{-1}.

(d) Now, prove Eq.(h) in p.339. You may prove the identity by explicitly working with the components, i.e.,

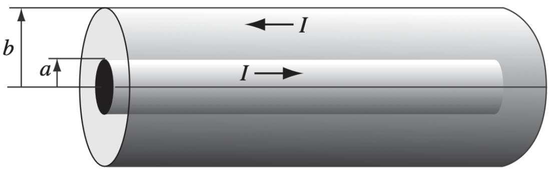
$$\nabla\cdot(\mathbf{U}\times\mathbf{V}) = \frac{\partial(\mathbf{U}\times\mathbf{V})_x}{\partial x} + \frac{\partial(\mathbf{U}\times\mathbf{V})_y}{\partial y} + \frac{\partial(\mathbf{U}\times\mathbf{V})_z}{\partial z} = ...$$

(e) Review how Ampere's law $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ is acquired from the Ampere's circuital law in Boas Chapter 6, Section 11, where $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density, and μ_0 is the permeability of free space.

(f) Using (d) and (e), show that the energy stored in magnetic fields is given by

$$W_\mathrm{m} = \frac{1}{2}\int(\mathbf{A}\cdot\mathbf{J})d\tau = \frac{1}{2\mu_0}\int|\mathbf{B}|^2 d\tau$$

for integration over all space, where $\mathbf{A}$ is the magnetic vector potential. You are first asked to briefly discuss how one arrived at the term $W_\mathrm{m} = \frac{1}{2}\int(\mathbf{A}\cdot\mathbf{J})d\tau$, and then explicitly prove the second equality. You may assume that $\mathbf{A}$ vanishes at large distance r at least as fast as r^{-1}.



[Adapted from Griffiths, Chapter 7.2]

(Note: The exercises here should sound familiar to most of you as you have begun to study and explore electromagnetism. If not, you may want to briefly review the classic textbooks in electromagnetism such as Griffiths or Jackson. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

• (c) One can think of $\frac{1}{2}\int\rho V d\tau$ as the continuum approximation of $W_\mathrm{e} = \frac{1}{2}\sum_i q_i V(\mathbf{r}_i)$ (see Section 2.4.2 of Griffiths for more information). Then, one gets

$$\begin{aligned} W_\mathrm{e} &= \frac{1}{2}\int \rho V d\tau = \frac{\epsilon_0}{2}\int V(\nabla\cdot\mathbf{E})d\tau = \frac{\epsilon_0}{2}\int\left[-\mathbf{E}\cdot(\nabla V) + \nabla\cdot(V\mathbf{E})\right]d\tau \\ &= \frac{\epsilon_0}{2}\left(\int|\mathbf{E}|^2 d\tau + \oint_{\partial\tau}(V\mathbf{E})\cdot\mathbf{n}d\sigma\right), \end{aligned}$$

where the second term vanishes as the integration is carried out over all space — since it behaves like $(r^{-1}r^{-2})\cdot r^2 = r^{-1}$ which tends to 0 as r approaches ∞.

• (f) One can think of $\frac{1}{2}\int(\mathbf{A}\cdot\mathbf{J})d\tau$ as the generalization of a energy stored in a current-carrying loop, $\frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})dl$, which itself came from $W_\mathrm{m} = \frac{1}{2}LI^2 = \frac{1}{2}I(\int\mathbf{B}\cdot d\mathbf{a}) = \frac{1}{2}I(\oint\mathbf{A}\cdot d\mathbf{l}) = \frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})dl$ (see Section 7.2.4 of Griffiths for more information). Then, one gets

$$\begin{aligned} W_\mathrm{m} &= \frac{1}{2}\int(\mathbf{A}\cdot\mathbf{J})d\tau = \frac{1}{2\mu_0}\int\mathbf{A}\cdot(\nabla\times\mathbf{B})d\tau = \frac{1}{2\mu_0}\int\left[\mathbf{B}\cdot(\nabla\times\mathbf{A}) - \nabla\cdot(\mathbf{A}\times\mathbf{B})\right]d\tau \\ &= \frac{1}{2\mu_0}\left(\int|\mathbf{B}|^2 d\tau + \oint_{\partial\tau}(\mathbf{A}\times\mathbf{B})\cdot\mathbf{n}d\sigma\right), \end{aligned}$$

where the second term vanishes as the integration is carried out over all space.

10. In the class we discussed the Legendre transformation as one of the examples of a simple change of variables that is found to be useful in classical mechanics and thermodynamics.

(a) Review the general discussion of a Legendre transformation in Boas Chapter 4, Section 11 by following the procedure step by step from Eq.(11.21) to Eq.(11.27).

(b) In one practical example in classical mechanics, given $L(x,\dot{x})$ with $dL = \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial \dot{x}}d\dot{x} = \frac{\partial L}{\partial x}dx + pd\dot{x}$ (note $p \equiv \frac{\partial L}{\partial \dot{x}}$), one can find $H(x,p)$ so that $dH = \frac{\partial L}{\partial x}dx - \dot{x}dp$. Find a Legendre transformation that gives $H(x,p)$. Discuss the meaning of the two functions, L and H, by identifying them as Lagrangian and Hamiltonian, respectively.

(c) In another example in thermodynamics which we briefly examined but left for your exercise, given $U(S,V)$ with $dU = \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial V}dV = TdS - PdV$ (note $T \equiv \left(\frac{\partial S}{\partial U}\right)^{-1}$ and $P \equiv -\frac{\partial U}{\partial V}$), one can find $H(S,P)$ so that $dH = TdS + VdP$. Find a Legendre transformation that gives $H(S,P)$. Discuss the meaning of the two functions, U and H, by identifying them as (internal) energy and enthalpy, respectively.

(d) Given $dU = TdS - PdV$, perform a different Legendre transformation to find another useful function related to energy, $F(T,V)$, known as the Helmholtz free energy. Finally, by performing Legendre transformations on both terms in dU, find yet another function related to energy, $G(T,P)$, known as the Gibbs free energy.

(Note: You may want to briefly review the textbooks in classical mechanics such as Thornton & Marion, and in statistical mechanics such as Schroeder or Reif. For (b) and (c), you are simply asked to come up with 2-3 sentences about how the these quantities are used in the respective textbooks, without going into all the mechanical or thermodynamical details.)

- (b) One can find $H(x,p) = -L + p\dot{x}$ as in Thornton & Marion Eq.(7.155) with $p \equiv \frac{\partial L}{\partial \dot{x}}$ in Eq.(7.151).
- (c) $H(S,P) = U + PV$ in Schroeder Eq.(1.51) with $T \equiv \left(\frac{\partial S}{\partial U}\right)^{-1}$ in Schroeder Eq.(3.4).
- (d) $F(T,V) = U - TS$ and $G(T,P) = U - TS + PV$ in Schroeder Eqs.(5.2)-(5.3).