Mathematical Physics I (Fall 2025): Homework #2 Solution

Due Oct. 3, 2025 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

1. Boas Chapter 3, Problem 5.36

(Note: For Problem 5.36, review the Examples in Boas Chapter 3, Section 5 in case you have not done it yet. Find the most relevant example and use the method depicted in there.)

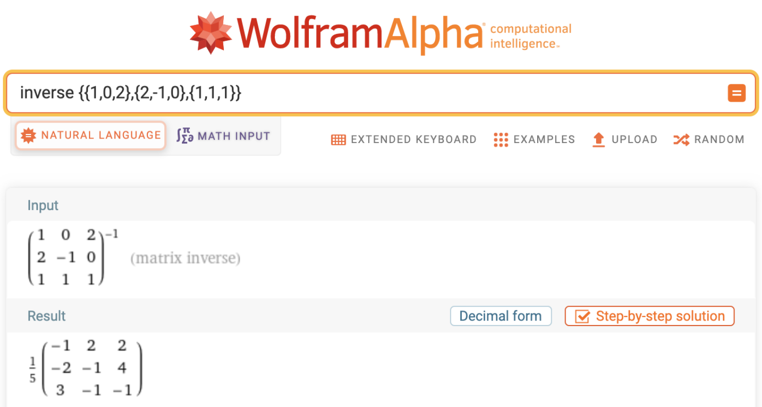
• The line found in Problem 5.11 is $\mathbf{r} = (4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + (\mathbf{i} - 2\mathbf{k})t$. To determine the distance between $P(3, 2, 5)$ and this line, let Q be a point on the line, say $Q(4, -1, 3)$ and $\mathbf{u}$ be the unit vector along the line. Then, from Figure 5.7 and the accompanying equations, the distance between P and the line is $|\overrightarrow{PQ} \times \mathbf{u}| = |(1, -3, -2) \times \frac{1}{\sqrt{5}}(1, 0, -2)| = |\frac{1}{\sqrt{5}}(6, 0, 3)| = 3$.

2. Boas Chapter 3, Problem 6.21

(Note: For Problem 6.21, find the inverse with two different methods — using Eq.(6.13) of Boas Chapter 3 and using the *Gauss-Jordan matrix inversion procedure*. Compare your results with the one found with a computer.)

• From Eq.(6.13),

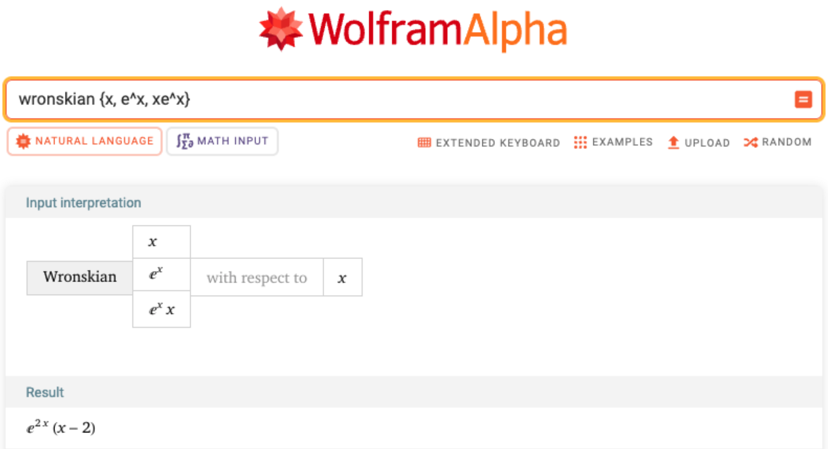
$$M = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \to \det(M) = 5 \text{ and } C = \begin{pmatrix} -1 & -2 & 3 \\ 2 & -1 & -1 \\ 2 & 4 & -1 \end{pmatrix} \to M^{-1} = \frac{1}{5}\begin{pmatrix} -1 & 2 & 2 \\ -2 & -1 & 4 \\ 3 & -1 & -1 \end{pmatrix}.$$



3. Boas Chapter 3, Problem 8.10

• Using the Wronskian in Eq.(8.5), you find

$$W = \begin{vmatrix} x & e^x & xe^x \\ 1 & e^x & e^x + xe^x \\ 0 & e^x & 2e^x + xe^x \end{vmatrix} = \begin{vmatrix} x & e^x & xe^x \\ 0 & \left(1-\frac{1}{x}\right)e^x & xe^x \\ 0 & e^x & 2e^x + xe^x \end{vmatrix} = x \begin{vmatrix} \left(1-\frac{1}{x}\right)e^x & xe^x \\ e^x & (2+x)e^x \end{vmatrix} = (x-2)e^{2x} \neq 0.$$



4. Boas Chapter 3, Problem 8.24

• Using Eq.(8.9),

$$\begin{vmatrix} 6-\lambda & 3 \\ 3 & -2-\lambda \end{vmatrix} = (\lambda-7)(\lambda+3) = 0,$$

and $\lambda_1 = 7$ gives the solution $\mathbf{r}_1 = (3\mathbf{i}+\mathbf{j})t$ while $\lambda_2 = -3$ gives $\mathbf{r}_2 = (\mathbf{i}-3\mathbf{j})s$.

5. Boas Chapter 3, Problem 9.17

• (c) $A = A^\dagger$ and $B = B^\dagger$ $\rightarrow$ from a complex counterpart of Eq.(9.11), $(AB)^\dagger = B^\dagger A^\dagger = BA = AB$ holds (i.e., AB is Hermitian) if and only if $[A, B] = 0$.

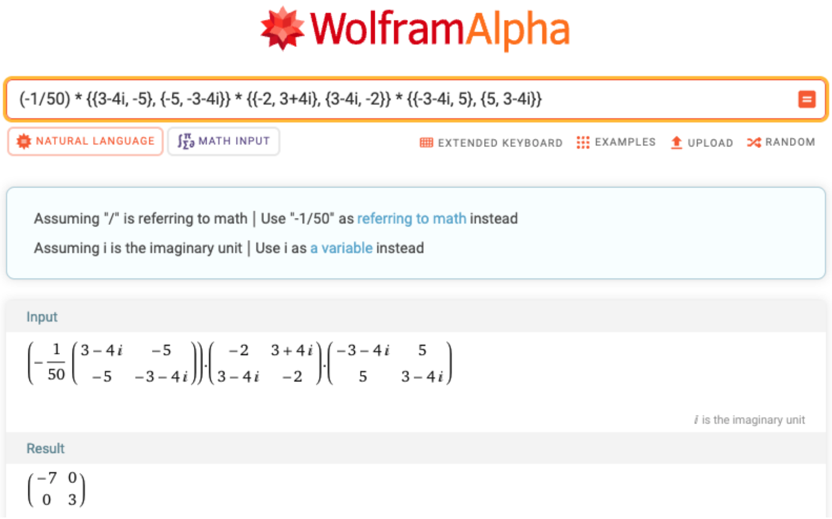
• (d) $A^{-1} = A^\dagger$ and $B^{-1} = B^\dagger$ $\rightarrow$ from Eq.(9.12), and from a complex counterpart of Eq.(9.11), $(AB)^{-1} = B^{-1}A^{-1} = B^\dagger A^\dagger = (AB)^\dagger$ holds (i.e., AB is unitary).

6. Boas Chapter 3, Problem 11.44

• From the characteristic equation, two eigenvalues $\lambda_1 = -7$ and $\lambda_2 = 3$ appear that correspond to eigenvectors $\mathbf{r}_1 = \frac{1}{5\sqrt{2}}\binom{-3-4i}{5}$ and $\mathbf{r}_2 = \frac{1}{5\sqrt{2}}\binom{5}{3-4i}$, respectively. Then it is straightforward to show that

$$U^{-1}HU = -\frac{1}{50}\begin{pmatrix} 3-4i & -5 \\ -5 & -3-4i \end{pmatrix}\begin{pmatrix} -2 & 3+4i \\ 3-4i & -2 \end{pmatrix}\begin{pmatrix} -3-4i & 5 \\ 5 & 3-4i \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

yields the diagonal matrix of eigenvalues.



7. Boas Chapter 3, Problem 11.60

(Note: For Problem 11.60, you will need to first prove then utilize the findings in Problem 11.57, or Eq.(11.36).)

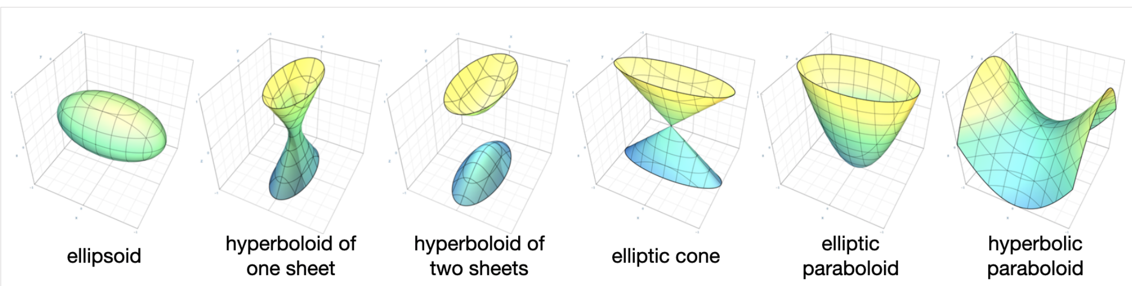
- With $C = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$ and $D = \begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ from Eq.(11.10),

$$\begin{aligned}C^{-1}(M^2 - 7M + 6I)C &= C^{-1}M^2C - 7C^{-1}MC + 6C^{-1}C\\ &= D^2 - 7D + 6I = \begin{pmatrix}\lambda_1^2 - 7\lambda_1 + 6 & 0\\ 0 & \lambda_2^2 - 7\lambda_2 + 6\end{pmatrix} = 0,\end{aligned}$$

which indicates that $M^2 - 7M + 6I = C(D^2 - 7D + 6I)C^{-1} = 0$.

8. Boas Chapter 3, Problem 12.9 (for 12.7)

(Note: For Problem 12.9, consider only Problem 12.7. Plot the surface using a computer, in both the original coordinate system and the principal axes coordinate system. Also, find the shortest distance from the origin to the surface, and identify the name of the surface. Examples of quadric surfaces in 3 dimensions include the following, as discussed in the class.)



- As in Example 2 of Boas Chapter 3, Section 12,

$$(x, y, z)\begin{pmatrix}1 & 2 & 2\\ 2 & 3 & 0\\ 2 & 0 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = 60 \;\to\; \begin{vmatrix}1-\lambda & 2 & 2\\ 2 & 3-\lambda & 0\\ 2 & 0 & 3-\lambda\end{vmatrix} = 0 = \{(1-\lambda)(3-\lambda) - 8\}(3-\lambda),$$

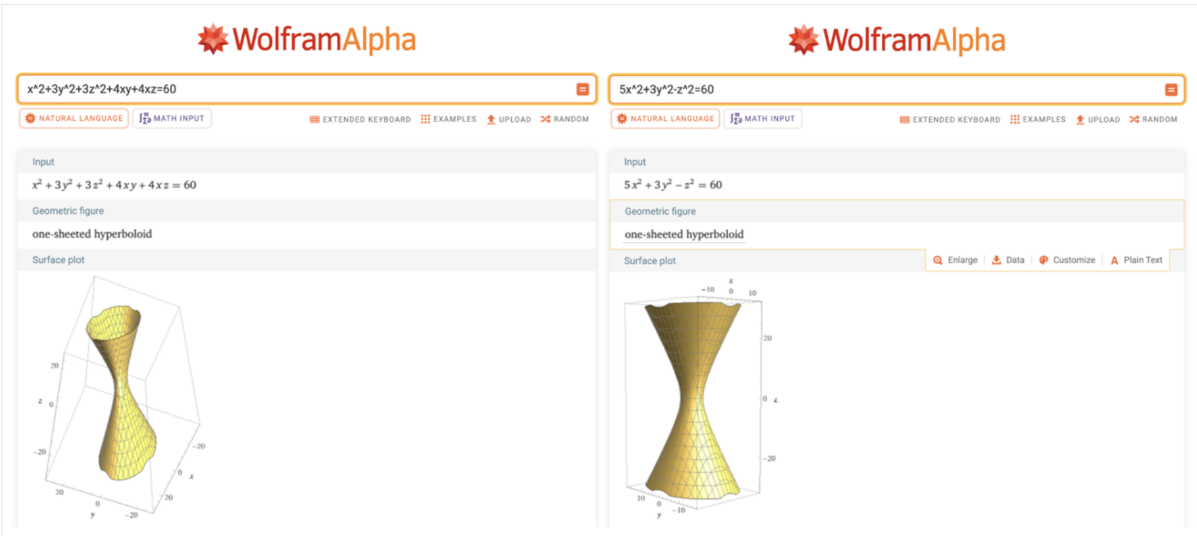
from which $\lambda = 5, 3, -1$ are acquired. Therefore, the new quadric surface equation relative to the principal axes becomes

$$(x', y', z') \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 60 \;\rightarrow\; \frac{x'^2}{12} + \frac{y'^2}{20} - \frac{z'^2}{60} = 1$$

which represents a hyperboloid of one sheet.[1] The corresponding eigenvectors $\mathbf{r}_1 = \frac{1}{\sqrt{3}}\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$, $\mathbf{r}_2 = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 0 \\ -1 \\ 1 \end{smallmatrix}\right)$ and $\mathbf{r}_3 = \frac{1}{\sqrt{6}}\left(\begin{smallmatrix} -2 \\ 1 \\ 1 \end{smallmatrix}\right)$ make up the rotation matrix C. Then it is straightforward to show that

$$C^{-1}MC = \frac{1}{6}\begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & -\sqrt{3} & \sqrt{3} \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & -2 \\ \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & \sqrt{3} & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

And finally, at $(x', y', z') = (\pm 2\sqrt{3}, 0, 0)$, the surface has the shortest distance to the origin, $d = 2\sqrt{3}$.



9. In this problem we consider the spin matrices in quantum mechanics that describe particles of various spins in three dimensions.

(a) First, work out Problem 6.6 in Boas Chapter 3. Here, for the Pauli spin matrices introduced to describe particles of spin 1/2,

$$A = \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

you will first need to show that $\sigma_j \sigma_k = \delta_{jk} I_2 + i \sum_l \epsilon_{jkl} \sigma_l$, where δ_{jk} and ϵ_{jkl} are defined in Eq.(9.4) of Boas Chapter 3 and in Eq.(5.3) of Chapter 10, respectively, and I_n is the $n \times n$ unit matrix. Then, it naturally follows that $[\sigma_j, \sigma_k] \equiv \sigma_j \sigma_k - \sigma_k \sigma_j = 2i \sum_l \epsilon_{jkl} \sigma_l$, which is called

[1] 일엽쌍곡면

the *fundamental commutation relation* for angular momentum matrices (or $[\sigma_j, \sigma_k] = 2i\sigma_l$ if $j, k, l = 1, 2, 3$ or a cyclic permutation thereof).

(b) Briefly discuss how these spin matrices are introduced and used in quantum mechanics. Prove that σ_x, σ_y and σ_z are both Hermitian and unitary. Show also that $\sigma^2 \equiv \sum_j \sigma_j^2 = 3I_2$.

(c) Now, using the 3×3 spin matrices that can describe particles of spin 1,

$$M_x = M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_y = M_2 = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_z = M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

show that $[M_j, M_k] = i \sum_l \epsilon_{jkl} M_l$ and $M^2 \equiv \sum_j M_j^2 = 2I_3$.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. For (b), you are simply asked to come up with 2-3 sentences about how the matrices are used, without diving into laborious quantum mechanical derivations. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

- (a) By combining $\sigma_j \sigma_k = i\sigma_l$ (if $j, k, l = 1, 2, 3$ or a cyclic permutation thereof) and $(\sigma_j)^2 = I_2$ (where I_2 is the 2×2 unit matrix), we can easily show $\sigma_j \sigma_k = \delta_{jk} I_2 + i \sum_l \epsilon_{jkl} \sigma_l$.

- (c) You may compare your answer from direct matrix multiplication with a computer solution.

SOURCE CODE

```
import numpy as np

A=np.multiply(1./np.sqrt(2), [[0,1,0],[1,0,1],[0,1,0]])
B=np.multiply(1./np.sqrt(2), [[0,-1j,0],[1j,0,-1j],[0,1j,0]])
C=[[1,0,0],[0,0,0],[0,0,-1]]

print("...Checking commutation relations...!")
print(np.dot(A,B)-np.dot(B,A))
print(np.multiply(1j, C))
print(np.dot(B,C)-np.dot(C,B))
print(np.multiply(1j, A))
print(np.dot(C,A)-np.dot(A,C))
print(np.multiply(1j, B))
print("")
print("...Checking M^2...!")
print(np.dot(A,A)+np.dot(B,B)+np.dot(C,C))
```

RESULTS

```
...Checking commutation relations...!
[[ 0.+1.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.-1.j]]
[[ 0.+1.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j -0.-1.j]]
[[ 0.+0.j          0.+0.70710678j  0.+0.j        ]
 [ 0.+0.70710678j  0.+0.j          0.+0.70710678j]
 [ 0.+0.j          0.+0.70710678j  0.+0.j        ]]
[[ 0.+0.j          0.+0.70710678j  0.+0.j        ]
 [ 0.+0.70710678j  0.+0.j          0.+0.70710678j]
 [ 0.+0.j          0.+0.70710678j  0.+0.j        ]]
[[ 0.          0.70710678  0.        ]
 [-0.70710678  0.          0.70710678]
 [ 0.         -0.70710678  0.        ]]
[[ 0.00000000+0.j  0.70710678+0.j  0.00000000+0.j]
 [-0.70710678+0.j  0.00000000+0.j  0.70710678+0.j]
 [ 0.00000000+0.j -0.70710678+0.j  0.00000000+0.j]]

...Checking M^2...!
[[ 2.+0.j  0.+0.j  0.+0.j]
 [ 0.+0.j  2.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  2.+0.j]]
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10. In the class we discussed the Gram-Schmidt orthonormalization process for a linear vector space and for a general vector space.

(a) In Example 6 of Boas Chapter 3, Section 14 — which we briefly discussed in class but left for you to work through — with an inner product defined as

$$\langle f|g\rangle = \int_{-1}^{1} f^*(x) g(x) dx,$$

one can start with the functions $f_i = x^i$ $(i = 0, 1, 2, 3)$ and construct a set of orthonormal polynomials P_i that satisfy the orthonormality condition on the interval $-1 \le x \le 1$,

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \delta_{mn}$$

(see also Eq.(8.4) of Chapter 12, but note a different normalization factor). We later identify this set of functions as the *Legendre polynomials.* Starting from Eq.(14.10), follow the procedure step by step and find for yourself the first four members of P_i.

(b) Consider a different set of functions defined with a similar inner product as in (a), but over a shifted interval $0 \le x \le 1$. The orthonormality condition is now given by

$$\int_{0}^{1} P'_m(x)P'_n(x)dx = \frac{1}{2m+1}\,\delta_{mn}.$$

Find the first three members in the set of orthonormal polynomials P'_i (the sign $'$ does not mean differentiation). We refer to this set of functions as the *shifted Legendre polynomials.*

(c) Now, with a new inner product defined as

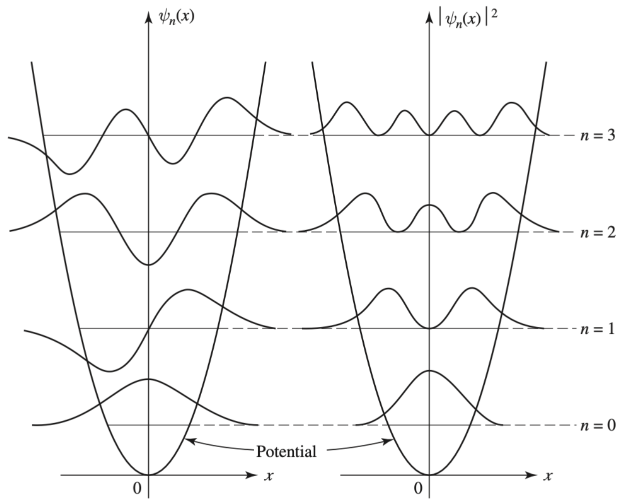
$$\langle f|g\rangle = \int_{-\infty}^{\infty} f^*(x)g(x)e^{-x^2}dx, \quad \text{where } e^{-x^2} \text{ is the “weighting” function,}$$

find the first three members in the set of orthonormal polynomials H_i that satisfy the orthonormality condition on the interval $-\infty \le x \le \infty$,

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2}dx = \delta_{mn}\sqrt{\pi}\,2^m m!$$

(see also Eq.(22.15) of Chapter 12). We refer to this set of functions as the *Hermite polynomials.*

(d) Discuss briefly where the (associated) Legendre polynomials and the Hermite polynomials appear in quantum mechanics or in other physics research.



[Adapted from Griffiths & Schroeter, Chapter 2.3]

(Note: For (c), you may simply use the value of the so-called Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ without proof, although it can be easily found using the techniques in Boas Chapter 5, Section

4, or in Problem 9.4 of Chapter 11. The Gaussian integral appears frequently also in elementary calculus textbooks such as Stewart (Chapter 15.3) or 김홍종 (Chapter 14.4). However, all other integrals such as $\int_{-\infty}^{\infty} x^2 e^{-x^2} dx$ must be evaluated explicitly, for example, by using integration by parts or the technique in Problem 12.16 of Boas Chapter 4. For (d), you may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. Once again, you are simply asked to come up with 2-3 sentences about how these polynomials are used, not the detailed physical or mathematical discussions. *Astronomy majors* are highly encouraged to look for the use cases of Legendre polynomials in astrophysics and cosmology.)

- (b-1) Following the notation in Example 6 of Boas Chapter 3, Section 14, but with $P'_i = e_i$, first we find $P'_0 = p_0 = f_0 = 1$ as $||p_0||^2 = \int_0^1 dx = 1$.

- (b-2) Then $p_1 = f_1 - P'_0 \int_0^1 P'_0 f_1 dx = x - 1 \cdot \int_0^1 x dx = x - \frac{1}{2} \;\rightarrow\;$ therefore $P'_1 = 2x - 1$, because $||p_1||^2 = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$ while $||P'_1||^2$ should be $\frac{1}{2 \cdot 1 + 1} = \frac{1}{3}$.

- (c-1) Again, following the notation in Example 6 of Boas Chapter 3, Section 14, but now with $H_i = e_i$, we find $H_0 = p_0 = f_0 = 1$ as $||p_0||^2 = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} = \delta_{00}\sqrt{\pi} \cdot 2^0 \cdot 0!$.

- (c-2) Then $p_1 = f_1 - H_0 \int_{-\infty}^{\infty} H_0 f_1 e^{-x^2} dx = x - 1 \cdot \int_{-\infty}^{\infty} x e^{-x^2} dx = x \;\rightarrow\;$ therefore $H_1 = 2x$, because $||p_1||^2 = \int_{-\infty}^{\infty} x^2 e^{-x^2} dx = \left[x \left(-\frac{1}{2} e^{-x^2} \right) \right]_{-\infty}^{\infty} + \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ while $||H_1||^2$ should be $\delta_{11}\sqrt{\pi} \cdot 2^1 \cdot 1! = 2\sqrt{\pi}$.

- (c-3) The list of the first few normalized Hermite polynomials are given in Eq.(22.13) of Chapter 12, or in p.804 (the answer to Problem 22.4 of Chapter 12).