Mathematical Physics I (Fall 2025): Homework #1 Solution

Due Sep. 19, 2025 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

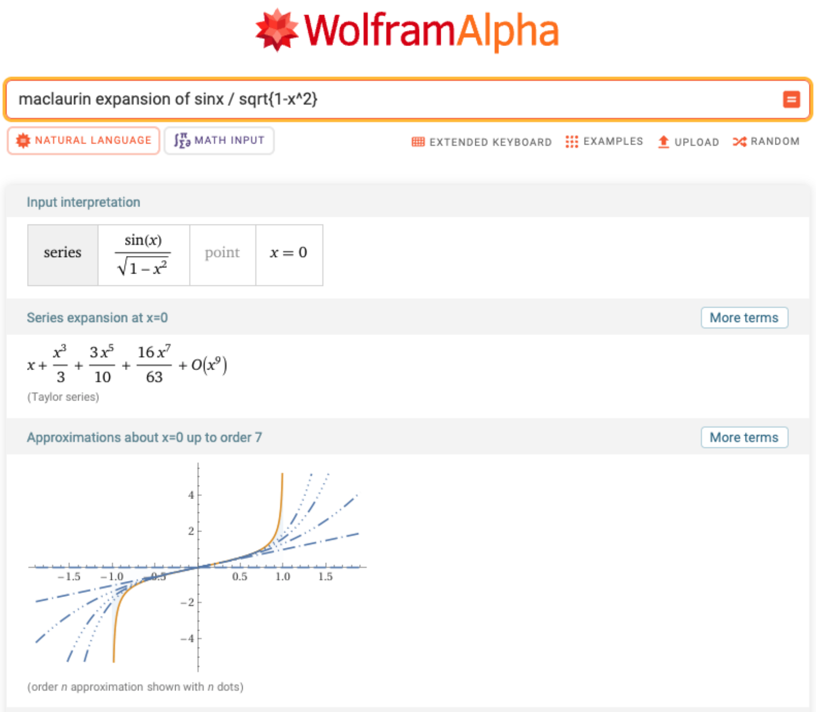
1. Boas Chapter 1, Problem 10.21

• The series converges for $0 \le x \le 1$ according to the ratio test.

2. Boas Chapter 1, Problem 13.36

(Note: For Problem 13.36, read the instruction in the textbook carefully; among others, you have been asked to compare your result with a computer solution as well.)

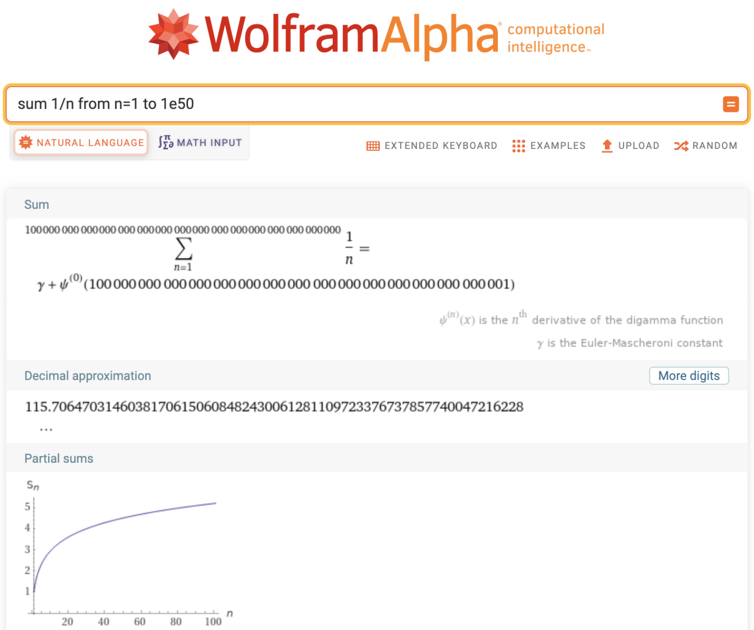
• From Eq.(13.1) and (13.5), you get the binomial series expansions, $\sin x \simeq x - \frac{x^3}{3!} + \frac{x^5}{5!}$ and $(1-x^2)^{\frac{1}{2}} \simeq 1 - (\frac{1}{2})x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^4$. Then, in a manner similar to Example 13.B2, the general terms of the Maclaurin series of the given integrand are found as $\frac{\sin x}{\sqrt{1-x^2}} \simeq x + \frac{1}{3}x^3 + \frac{3}{10}x^5$. Integrating both sides, you find $\int_0^u \frac{\sin x}{\sqrt{1-x^2}}dx \simeq \frac{1}{2}u^2 + \frac{1}{12}u^4 + \frac{1}{20}u^6$.



3. Boas Chapter 1, Problem 16.1

(Note: For Problem 16.1, it is helpful to draw a diagram that considers the balance between forces and torques, a so-called *extended free-body diagram.*)

• You may find it interesting to read an article about the stacking problem such as Hall, J., 2005, "Fun with stacking block", American Journal of Physics, 73, 1107.



4. Boas Chapter 2, Problem 10.23

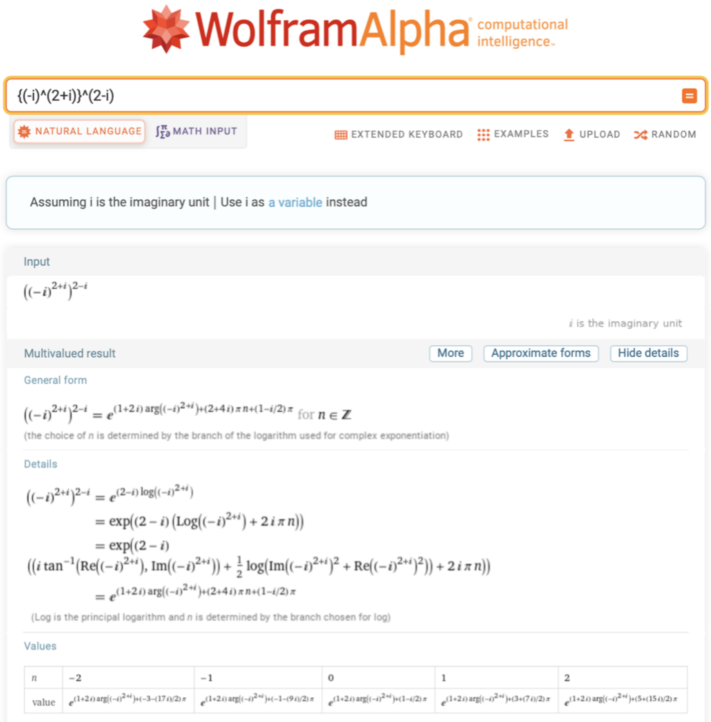
• $\left\{16\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\right\}^{\frac{1}{4}} = \left(16e^{\frac{2\pi}{3}+2n\pi}\right)^{\frac{1}{4}} = 2e^{\frac{\pi}{6}+\frac{n\pi}{2}} = \{\sqrt{3}+i,\, -1+i\sqrt{3},\, -\sqrt{3}-i,\, 1-i\sqrt{3}\}.$

5. Boas Chapter 2, Problem 14.24

(Note: For Problem 14.24, compare your result with a computer solution. Resolve or explain any disagreement in the two results.)

• (a-1) Using the definition of complex powers in Eq.(14.1), $\left[(-i)^{2+i}\right]^{2-i} = \left[e^{(2+i)\ln(-i)}\right]^{2-i} = \left[e^{(2+i)i\left(\frac{3\pi}{2}+2n\pi\right)}\right]^{2-i} = \left[e^{2i\left(\frac{3\pi}{2}+2n\pi\right)} \cdot e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]^{2-i} = \left[-e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]^{2-i} = e^{(2-i)\ln\left[-e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]} = e^{(2-i)\left[i(\pi+2n\pi)+\ln\left[e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]\right]} = e^{2\ln\left[e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]+(\pi+2n\pi)} \cdot e^{i\left(-\ln\left[e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]+2(\pi+2n\pi)\right)}.$

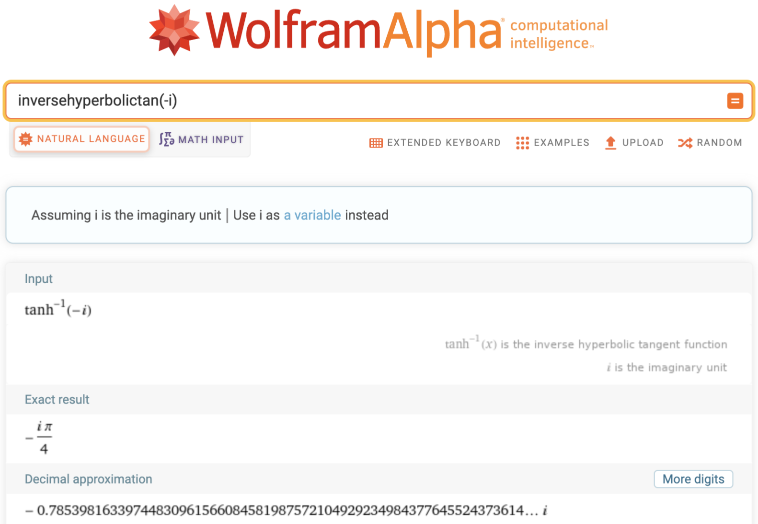
• (a-2) Meanwhile, $(-i)^{(2+i)(2-i)} = -i$, which can be confirmed with the computer solution below. Note that, from our analytic calculations and computer solutions, your answer in (a-2) is a special case in your answer in (a-1).



6. Boas Chapter 2, Problem 15.6

(Note: For Problem 15.6, read the instruction in the textbook carefully; that is, you have been asked to compare your result with a computer solution as well. Resolve or explain any disagreement in results.)

- With $u \equiv e^z$, you get $\tanh z = -i = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{u - u^{-1}}{u + u^{-1}} \rightarrow u^2 = \frac{1-i}{1+i} = -i \rightarrow z = \ln u = \ln\left[\pm(-i)^{\frac{1}{2}}\right] = \ln\left[e^{i(\frac{3}{2}\pi + 2n\pi)\cdot\frac{1}{2}}\right] = i(n\pi + \frac{3}{4}\pi)$.



7. Boas Chapter 2, Problem 17.21

(Note: For Problem 17.21, verify only the first equality, $\cosh^{-1} z = \ln\left(z \pm \sqrt{z^2-1}\right)$.)

• $\cosh^{-1} z = \omega \;\rightarrow\;$ With $u \equiv e^{\omega}$, you find $z = \cosh\omega = \frac{e^{\omega}+e^{-\omega}}{2} = \frac{u+u^{-1}}{2} \;\rightarrow\; u^2 - 2zu + 1 = 0$ $\rightarrow\;\; \omega = \ln u = \ln(z \pm \sqrt{z^2-1}) = \cosh^{-1} z$.

8. Boas Chapter 3, Problem 2.8

(Note: For Problem 2.8, again, read the instruction in the textbook carefully.)

• The rank of the augmented matrix is 2 while there are 3 unknowns.

9. In the quantum statistical distribution of bosons with unspecified total number of particles (so-called Bose-Einstein statistics of quantized oscillators), the average energy of the system is found to be

$$\bar{\epsilon} = \frac{\sum\limits_{n=1}^{\infty} n\epsilon_0 e^{-\frac{n\epsilon_0}{k_B T}}}{\sum\limits_{n=0}^{\infty} e^{-\frac{n\epsilon_0}{k_B T}}}, \tag{1}$$

where ϵ_0 is a fixed energy, k_B is the Boltzmann constant, and T is the temperature. (a) After briefly discussing how this formula is acquired, show that the ratio becomes

$$\bar{\epsilon} = \frac{\epsilon_0}{e^{\frac{\epsilon_0}{k_B T}} - 1}$$

by first identifying Eq.(1)'s denominator as a binomial expansion of $\frac{1}{1-x}$ with $x = e^{-\frac{\epsilon_0}{k_B T}}$, and its numerator as a constant times $\frac{x}{(1-x)^2}$. (b) Using a power series expansion, show that $\bar{\epsilon}$ reduces to $k_B T$ in the classical limit of $k_B T \gg \epsilon_0$.

(Note: You may want to briefly review the classic textbooks in statistical mechanics such as Schroeder or Reif. To receive full credit, you are asked to come up with a short paragraph about what Eq.(1) means, without having to laboriously derive the equation in great detail. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

- (a) By differentiating $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, you find $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n$, which leads to

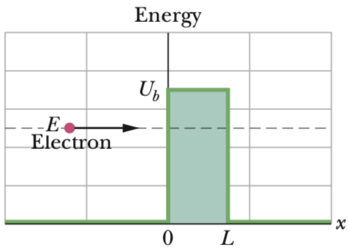
$$\bar{\epsilon} = \frac{\sum_{n=1}^{\infty} \epsilon_0 n x^n}{\sum_{n=0}^{\infty} x^n} = \frac{\epsilon_0 x}{1-x} = \frac{\epsilon_0}{x^{-1}-1}.$$

- (b) With $y = \frac{\epsilon_0}{k_B T} \ll 1$, you get $\bar{\epsilon} = \frac{\epsilon_0}{e^y - 1} \simeq \frac{\epsilon_0}{1+y-1} = k_B T$.

10. In quantum mechanics, consider the transmission coefficient T of an electron matter wave (of mass m and energy E) incident on a rectangular potential barrier of height U_b (see figure). Briefly explain why the following expression must be evaluated to determine the probability of the electron being transmitted through the barrier and detected on the other side:

$$T^{-1} = \left| \frac{(k+i\kappa)^2 - (k-i\kappa)^2 e^{-2\kappa L}}{4ik\kappa e^{-\kappa L} e^{-ikL}} \right|^2 = 1 + \left(\frac{k^2+\kappa^2}{2k\kappa} \right)^2 \sinh^2(\kappa L), \tag{2}$$

where $k \equiv \frac{\sqrt{2mE}}{\hbar}$ and $\kappa \equiv \frac{\sqrt{2m(U_b - E)}}{\hbar}$ are real numbers. Prove the second equality using complex algebra.



[Adapted from Halliday & Resnick, Chapter 38.9]

(Note: You may want to review the freshman physics textbooks such as Halliday & Resnick (Chapters 38.8 and 38.9), or the quantum mechanics textbooks such as Griffiths & Schroeter

(Chapter 2.6 and problems therein) or Landau & Lifshitz. To receive full credit, you should include a brief explanation — without a detailed derivation — of how the absolute square of a complex number like Eq.(2) arises in calculating the transmission probability.)

- The wave function $\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx}, & \text{for } x < 0, \\ Ce^{-\kappa x} + De^{\kappa x}, & \text{for } 0 < x < L, \\ Fe^{ikx}, & \text{for } x > L \end{cases}$ describes an electron in the three regions in the figure. Using the four boundary conditions – continuity of ψ and $\frac{d\psi}{dx}$ at $x = 0$ and L – on five unknowns, one can get the ratios $\frac{D}{C}$, $\frac{B}{A}$, and $\frac{F}{A}$. In particular, one finds the inverse of the transmission coefficient as

$$T^{-1} = \left|\frac{A}{F}\right|^2 = \left|\frac{(k+i\kappa)^2 - (k-i\kappa)^2 e^{-2\kappa L}}{4ik\kappa e^{-\kappa L} e^{-ikL}}\right|^2.$$

- Now with complex algebra,

$$\begin{aligned}
T^{-1} &= \left|\frac{(k+i\kappa)^2 e^{\kappa L} - (k-i\kappa)^2 e^{-\kappa L}}{4ik\kappa e^{-ikL}}\right|^2 = \left|\frac{(k^2-\kappa^2+2ik\kappa)e^{\kappa L} - (k^2-\kappa^2-2ik\kappa)e^{-\kappa L}}{4ik\kappa e^{-ikL}}\right|^2 \\
&= \left|\frac{2(k^2-\kappa^2)\sinh(\kappa L) + 4ik\kappa\cosh(\kappa L)}{4ik\kappa e^{-ikL}}\right|^2 = \frac{(k^2-\kappa^2)^2\sinh^2(\kappa L) + 4k^2\kappa^2\cosh^2(\kappa L)}{4k^2\kappa^2} \\
&= \frac{(k^2+\kappa^2)^2\sinh^2(\kappa L) + 4k^2\kappa^2\left\{\cosh^2(\kappa L) - \sinh^2(\kappa L)\right\}}{4k^2\kappa^2} \\
&= 1 + \left(\frac{k^2+\kappa^2}{2k\kappa}\right)^2 \sinh^2(\kappa L).
\end{aligned}$$