

Mathematical Physics I (Fall 2025): Homework #5

Due Nov. 21, 2025 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

- By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)
- Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.
- Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.

1.-8. Boas Chapter 8, Problems 5.6, 6.6, 6.23, 6.35, 7.3, 7.8, 8.10, 9.9

(Note: For Problems in Sections 5 and 6, read the instruction in the textbook carefully; that is, you have been asked to find a computer solution and reconcile differences, if any. You may continue to utilize computer solutions to validate your answers to problems in Sections 7 to 9. For Problem 7.8, you may find it useful to first review Case (c) and Example 2 of Boas Chapter 8, Section 7. Then, approach the problem purely mathematically, and only after obtaining your solution, verify whether it is consistent with the energy conservation argument from classical dynamics. For Problem 8.10, solve it in two different ways, as suggested in the provided hint.)

9. At the start of Boas Chapter 8, Section 7, the author discusses several methods for solving various types of second-order ODEs. Among them is Lagrange's *method of variation of parameters* to find a particular solution of an inhomogeneous ODE.

(a) Let us begin with a homogeneous second-order linear ODE in the form of

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous functions of x . Let us assume that we know its two independent solutions, y_1 and y_2 . Now, for the inhomogeneous second-order linear ODE of

$$y'' + p(x)y' + q(x)y = f(x),$$

show that a particular solution $y_p(x)$ is written as

$$y_p(x) = -y_1(x) \int \frac{y_2(x')f(x')}{W(x')}dx' + y_2(x) \int \frac{y_1(x')f(x')}{W(x')}dx'$$

where $W(x')$ is the Wronskian of y_1 and y_2 , $W(y_1(x'), y_2(x'))$.

(Note: You may start with $y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$ and follow the step-by-step instruction given in Boas Chapter 8, Problem 12.14(b) that leads to the set of two conditions for c_1 and c_2 : $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ and $c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = f(x)$. Notice that the first equation of this set is our “imposed” condition, while the second one is what you get if you plug $y_p' = \{c_1'(x)y_1(x) + c_2'(x)y_2(x)\} + \{c_1(x)y_1'(x) + c_2(x)y_2'(x)\} = c_1(x)y_1'(x) + c_2(x)y_2'(x)$ and the corresponding y_p'' into our ODE above. In case you wonder, no knowledge about the Green function in Section 12 is needed to tackle this problem.)

Now, utilizing the given solution of the homogeneous equation, find a solution of each of the following inhomogeneous ODEs. (More exercise problems in Chapter 8, Problems 12.15-18.)

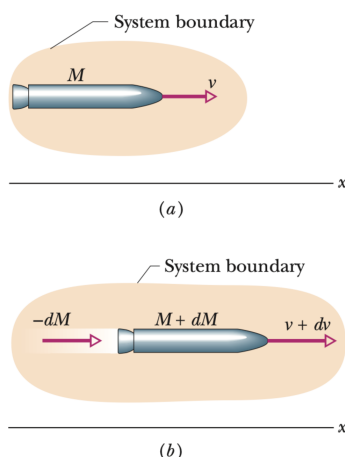
(b) $y'' + y = \sec x$; with $y_1 = \cos x$ and $y_2 = \sin x$

(c) $y'' - 3y' + 2y = \sin x$; with $y_1 = e^x$ and $y_2 = e^{2x}$

10. To practice formulating differential equations of motion, let us consider a rocket of variable mass m , initially at rest with mass m_0 . The rocket is propelled by steadily ejecting fuel at speed $u > 0$ relative to the rocket (i.e., the exhaust speed measured in the rocket’s frame is constant).

(a) Neglecting gravity, show that the rocket’s equation of motion can be written as $\frac{dv}{dm} = -\frac{u}{m}$. Find v as a function of m .

(b) Now, consider the vertical ascent in the presence of uniform gravitational field g acting downward. Show that the rocket’s equation of motion becomes $\frac{dv}{dm} = \frac{g}{\alpha} - \frac{u}{m}$ with the constant fuel ejection rate $\alpha = -\frac{dm}{dt} > 0$. Find v as a function of m .



[Adapted from Halliday & Resnick, Chapter 9.9]

(Note: For the dynamics of a rocket with variable mass, you may want to review the freshman physics textbooks such as Halliday & Resnick (Chapter 9.9), or the classical mechanics textbooks such as Thornton & Marion (Chapter 9.11).)

[Problem 10 continues in the next page.]

(c) Finally, in the relativistic regime where $\frac{v}{c}$ is non-negligible (but assume $\frac{u}{c}$ is negligible so the exhaust speed is still small compared to the speed of light c), show that, when gravity is ignored, the rocket's equation of motion is written as $\frac{dv}{dm} = -\frac{u}{m} \left(1 - \frac{v^2}{c^2}\right)$, where v is the relative speed of the rocket and the lab (inertial) frame. Find v as a function of m .

(Note: To receive full credit for (c), you should briefly explain how the factor $\left(1 - \frac{v^2}{c^2}\right)$ arises in the equation of motion. To do so, you may start from the relativistic velocity addition formula in Halliday & Resnick (Chapter 37.4) or in Thornton & Marion (Chapter 14.9), set $v_1 = v(m)$ and $v_2 = dv$, and expand $v(m+dm)$ to first order in dm .¹ When integrating the resulting differential equation, you may prove and utilize the indefinite integral $\int \frac{dz}{1-z^2} = \ln \sqrt{\frac{1+z}{1-z}} = \tanh^{-1} z$, seen in Problem 13.14 of Boas Chapter 1, or in common integral tables like Appendix E of Thornton & Marion, or in extensive references like Zwillinger (Section 5.4.3, 33rd ed.). To evaluate the integral explicitly, you may need to perform a decomposition $\frac{1}{1-z^2} = \frac{1}{2} \left(\frac{1}{1+z} + \frac{1}{1-z} \right)$.)

¹To be precise, the rocket here can be described as accelerating with a constant *proper* acceleration. The *proper* acceleration is defined as follows: Let t be the time coordinate in the rocket's frame. If the *proper* acceleration is a , then at time $t + dt$, the rocket moves at speed adt relative to the instantaneous inertial frame it occupied at time t . For a more formal treatment, see advanced texts on special relativity.