Mathematical Physics I (Fall 2025): Final Exam Solution

Dec. 13, 2025

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

1. (a) [2 pt] Find the exponential Fourier transform and the sine transform of

$$f(x) = \begin{cases} 1, & -2 < x < 0, \\ -1, & 0 < x < 2, \\ 0 & \text{otherwise}, \end{cases}$$

and use your result to evaluate

$$\int_0^\infty \frac{\sin^3 \alpha}{\alpha}\, d\alpha\,.$$

(Note: For the latter question, you may want to recall the trigonometric identity $\frac{1-\cos 2y}{2} = \sin^2 y$.)

(b) [2 pt] The periodic function $f(x) = |x|$ is given over one period $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Sketch several periods of $f(x)$. Expand $f(x)$ in an appropriate Fourier series, and use your result to evaluate

$$\sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

- (a) From Eq.(12.14) of Boas Chapter 7, you can find

$$g_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \alpha x\, dx = -\sqrt{\frac{2}{\pi}} \int_0^2 \sin \alpha x\, dx = \sqrt{\frac{2}{\pi}} \cdot \frac{\cos 2\alpha - 1}{\alpha},$$

which gives

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g_s(\alpha) \sin \alpha x\, d\alpha = \frac{2}{\pi} \int_0^\infty \frac{(\cos 2\alpha - 1) \sin \alpha x}{\alpha}\, d\alpha.$$

Meanwhile, from Eq.(12.2) of Boas Chapter 7, you find

$$g(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-i\alpha x}\,dx = \frac{1}{2\pi}\left\{\int_{-2}^{0} e^{-i\alpha x}\,dx - \int_{0}^{2} e^{-i\alpha x}\,dx\right\}$$
$$= \frac{1}{2\pi}\cdot\frac{e^{2i\alpha}+e^{-2i\alpha}-2}{i\alpha} = \frac{\cos 2\alpha - 1}{i\pi\alpha},$$

which eventually gives the identical $f(x)$ as above:

$$f(x) = \int_{-\infty}^{\infty} g(\alpha)e^{i\alpha x}\,d\alpha = \int_{-\infty}^{\infty}\frac{\cos 2\alpha - 1}{i\pi\alpha}e^{i\alpha x}\,d\alpha$$
$$= \frac{1}{\pi}\int_0^{\infty}\frac{\cos 2\alpha - 1}{i\alpha}\left(e^{i\alpha x} - e^{-i\alpha x}\right)d\alpha = \frac{2}{\pi}\int_0^{\infty}\frac{(\cos 2\alpha - 1)\sin\alpha x}{\alpha}\,d\alpha.$$

Then, from the Fourier integral theorem in Boas Chapter 7, Section 12, you find

$$-1 = f(1) = \frac{2}{\pi}\int_0^{\infty}\frac{(\cos 2\alpha - 1)\sin\alpha}{\alpha}\,d\alpha = \frac{2}{\pi}\int_0^{\infty}\frac{(-2\sin^2\alpha)\sin\alpha}{\alpha}\,d\alpha \quad\rightarrow\quad \int_0^{\infty}\frac{\sin^3\alpha}{\alpha}\,d\alpha = \frac{\pi}{4}.$$

• (b) From Eqs.(5.6), (5.9)-(5.10) or (9.5) of Boas Chapter 7, you can find $b_n = 0$,

$$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}|x|dx = \frac{4}{\pi}\int_0^{\pi/2} x\,dx = \frac{\pi}{2},$$

and

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} x\cos 2nx\,dx = \frac{4}{\pi}\left\{\left[\frac{x\sin 2nx}{2n}\right]_0^{\pi/2} - \frac{1}{2n}\int_0^{\pi/2}\sin 2nx\,dx\right\}$$
$$= \frac{(-1)^n - 1}{n^2\pi} = \begin{cases} 0, & \text{if } n \text{ even,}\\ -\frac{2}{n^2\pi}, & \text{if } n \text{ odd.}\end{cases}$$

Then from Parseval's theorem in Eq.(11.4) of Boas Chapter 7, you find

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}|x|^2\,dx = \frac{\pi^2}{12} = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}a_n^2 = \frac{\pi^2}{16} + \frac{2}{\pi^2}\sum_{\substack{n=1\\ n\,\text{odd}}}^{\infty}\frac{1}{n^4} \quad\rightarrow\quad \sum_{\substack{n=1\\ n\,\text{odd}}}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{96}.$$

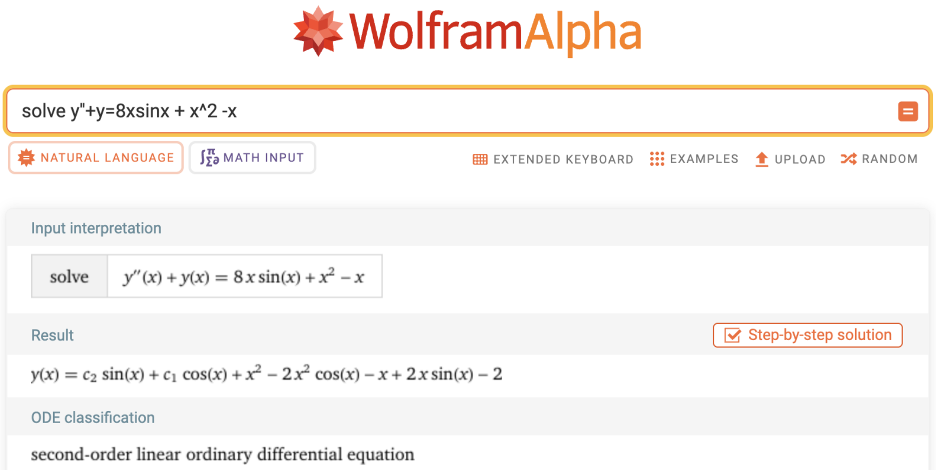
2. Find the general solution to each of the following differential equations.

(a) [2 pt] $(D^2+1)y = 8x\sin x + x^2 - x$

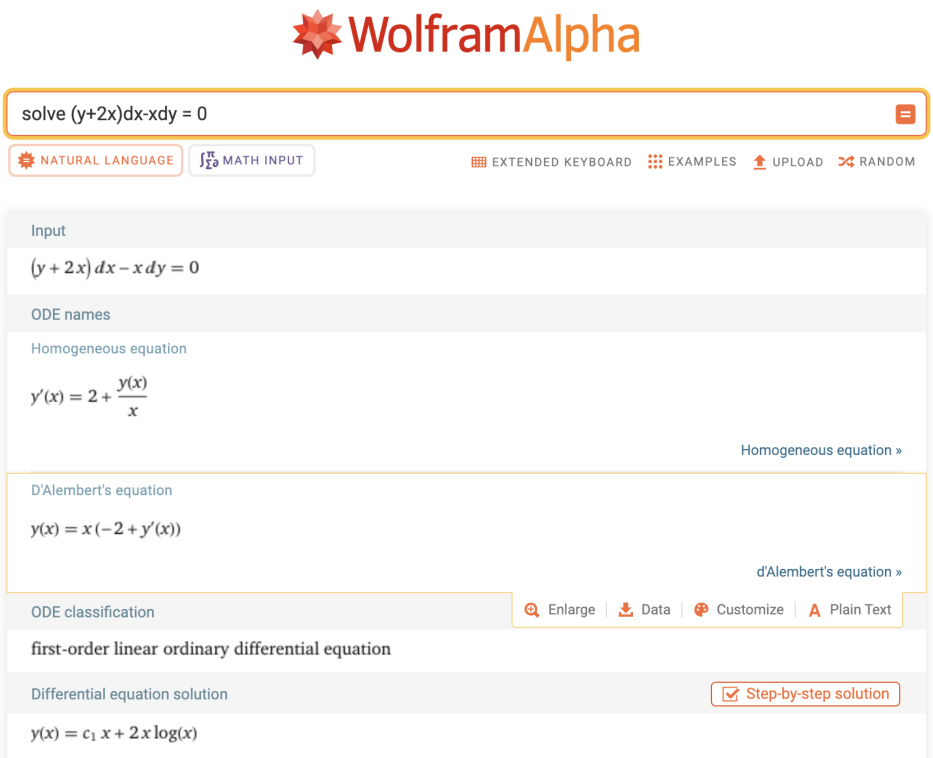
(b) [1 pt] $(y+2x)dx - xdy = 0$

• (a-1) The homogeneous equation $y'' + y = 0$ gives the complementary solution $y_c = Ae^{ix} + Be^{-ix}$. And from Eq.(6.24) of Boas Chapter 8, the inhomogeneous equation $y'' + y = x^2 - x$ requires us try a particular solution of the form $y_{p1} = C_1x^2 + C_2x + C_3$. Plugging y_{p1} back into the equation yields $C_1 = 1$, $C_2 = -1$, and $C_3 = -2$.

• (a-2) Now to find a particular solution y_{p2} for $y'' + y = 8x\sin x$, we will need find a particular solution of the equation $y'' + y = 8xe^{ix}$, using the technique in Examples 6 and 7 of Boas Chapter 8, Section 6. Because i equals to one of the roots of the auxiliary equation, from Eq.(6.24) you try a particular solution of the form $Y_{p2} = xe^{ix}(C_4x + C_5)$. Plugging Y_{p2} back into the equation yields $C_4 = -2i$, $C_5 = 2$ and $Y_{p2} = xe^{ix}(-2ix + 2)$. Therefore, the particular solution we need is $y_{p2} = \text{Im}(Y_{p2}) = -2x^2\cos x + 2x\sin x$. Combining all the above, you reach $y = y_c + y_{p1} + y_{p2} = Ae^{ix} + Be^{-ix} + (x^2 - x - 2) - 2x^2\cos x + 2x\sin x$.

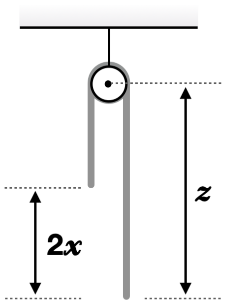


• (b) Rearranging the equation given, you get $y' - \frac{1}{x}y = 2$. Using the technique in Eq.(3.9) of Boas Chapter 8, Section 3, $I = \int P dx = -\int \frac{dx}{x} = -\ln x \;\rightarrow\; y = e^{-I}\int Qe^{I}dx + Ce^{-I} = x\int 2\cdot\frac{dx}{x} + Cx = 2x\ln x + Cx$.



3. (a) [2 pt] A flexible chain of total length l hangs over a frictionless peg of negligible size, with one end of the chain slightly longer than the other (see figure). Let $2x$ denote the difference in length between the two ends of the chain and take $x = x_0$ when $t = 0$. Assuming that the chain is released from rest and slides off without friction, write down the differential equation of motion using the variable x. Show that, for $0 < x < \frac{l}{2}$,

$$x(t) = x_0 \cosh\left(t\sqrt{\frac{2g}{l}}\right).$$



(b) [2 pt] Show that the Lagrangian of a particle of rest mass m in a motion along x-axis,

$$L = mc^2(1 - \sqrt{1 - v^2/c^2}) - V(x),$$

leads to a relativistic equation of motion,

$$F = \frac{d}{dt}\frac{mv}{\sqrt{1 - v^2/c^2}}.$$

Here, c is the speed of light, t and $v = \frac{dx}{dt}$ are the ordinary time and velocity, respectively. Then, consider a particle subject to a constant force F. By integrating the separable equation, determine $v(t)$. Verify that $v \to c$ as $t \to \infty$. Finally, compute the distance $s(t)$ traveled by the particle starting from rest. (Note: m is constant. If you encounter an integral of the form $\int \frac{dy}{(1-y^2)^{3/2}}$, consider the substitution $y = \sin\theta$.)

- (a) The Newtonian equation of motion is written as

$$\rho l \frac{d^2 z}{dt^2} = \rho l \ddot{z} = \rho z g - \rho(l - z)g = \rho(2z - l)g = \rho(2x)g,$$

where ρ is the line density of the chain. Here, since $2x = 2z - l$, you can replace $\ddot{z}$ with $\ddot{x}$ to get

$$\rho l \ddot{x} = \rho(2x)g \quad \rightarrow \quad \ddot{x} - \frac{2g}{l}x = 0 \quad \rightarrow \quad x = Ae^{t\sqrt{2g/l}} + Be^{-t\sqrt{2g/l}},$$

where the coefficients can be found from $x(0) = x_0$ and $\dot{x}(0) = 0$, yielding $x(t) = x_0 \cosh\left(t\sqrt{2g/l}\right)$.

- (b-1) From Eq.(5.3) of Boas Chapter 9, with an independent variable t, you can find

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \rightarrow -\frac{\partial V}{\partial x} - \frac{d}{dt}\left[\frac{-mc^2\left(-\frac{2v}{c^2}\right)\cdot\frac{1}{2}}{\sqrt{1 - v^2/c^2}}\right] = F - \frac{d}{dt}\frac{mv}{\sqrt{1 - v^2/c^2}} = 0.$$

Note that $\mathbf{F} = -\nabla V$ is used, as in Example 2 of Boas Chapter 9, Section 5.

- (b-2) By explicitly carrying out the differentiation, you get

$$F = m\frac{d}{dt}\frac{v}{\sqrt{1-v^2/c^2}} = m\cdot\frac{\frac{dv}{dt}\sqrt{1-v^2/c^2} - v\cdot\frac{(-2v/c^2)}{2\sqrt{1-v^2/c^2}}}{1-v^2/c^2} = m\frac{dv}{dt}\frac{1}{(1-v^2/c^2)^{\frac{3}{2}}}.$$

Then one can separate the variables, and perform the integration with $\frac{v}{c} = \sin\theta$ as

$$\int_0^t \frac{F}{m}dt = \int_0^v \frac{dv}{(1-v^2/c^2)^{\frac{3}{2}}} = \int_0^\theta \frac{c\cos\theta d\theta}{(1-\sin^2\theta)^{\frac{3}{2}}} = c\int_0^\theta \sec^2\theta d\theta = c\tan\theta = \frac{cv}{\sqrt{c^2-v^2}} = \frac{Ft}{m},$$

which yields

$$v = v(t) = \frac{Fct}{\sqrt{m^2c^2+F^2t^2}}.$$

The behavior of $v(t)$ at $t\to\infty$ can be seen by slightly rearranging $v(t)$ as

$$v(t) = \frac{c}{\sqrt{\left(\frac{mc}{Ft}\right)^2+1}} \to c \quad \text{as} \quad t\to\infty.$$

Finally, integrating $v(t)$ above, you find

$$s(t) = \int_0^t v(t)dt = \int_0^t \frac{Fct\,dt}{\sqrt{m^2c^2+F^2t^2}} = \frac{c}{F}\left[\sqrt{m^2c^2+F^2t^2}\right]_0^t = \frac{c}{F}\left[\sqrt{m^2c^2+F^2t^2} - mc\right].$$

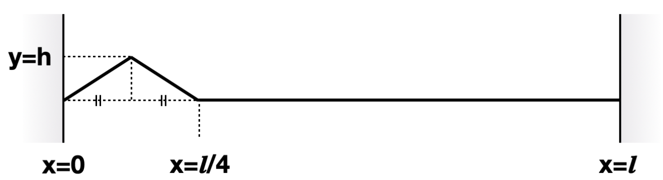
4. (a) [2 pt] A mechanical system is described by the differential equation $y'' + 4y = f(t)$ with

$$f(t) = \begin{cases} 1, & \text{if } 0 < t < a, \\ 0, & \text{if } t > a. \end{cases}$$

Assuming $y_0 = y_0' = 0$ at $t = 0$ and using the Laplace transform, find the response $y(t)$ for $t > 0$. Sketch the motion of the system for two cases: *(i)* $a = \frac{3}{2}\pi$ and *(ii)* $a = \frac{13}{4}\pi$. (Note: The table of Laplace transforms is in the last page of this exam.)

(b) [1 pt] Repeat (a), this time using the Green function. (Note: To utilize the method of Green function, you will first need to explicitly derive the response of this system to a unit impulse at time $t = t'\ (> 0)$. Again, the table of Laplace transforms is in the last page of this exam.)

(c) [2 pt] A string of length l is initially at rest and given the triangular displacement y_0 shown below. Assuming the string is fixed at both ends, determine the displacement y as a function of position x, time t, and the wave velocity v (a constant determined by the tension and the linear density of the string). For an additional +1 point, repeat the problem under the new boundary conditions where the string is pinned at $x = 0$ but is free to move up and down at $x = l$.



• (a) With Eqs.(9.1)-(9.2) of Boas Chapter 8, you find

$$y'' + 4y = f(t) \quad \rightarrow \quad p^2 Y + 4Y = \int_0^\infty f(t) e^{-pt} dt = \int_0^a e^{-pt} dt = \frac{1}{p}(1 - e^{-pa})$$

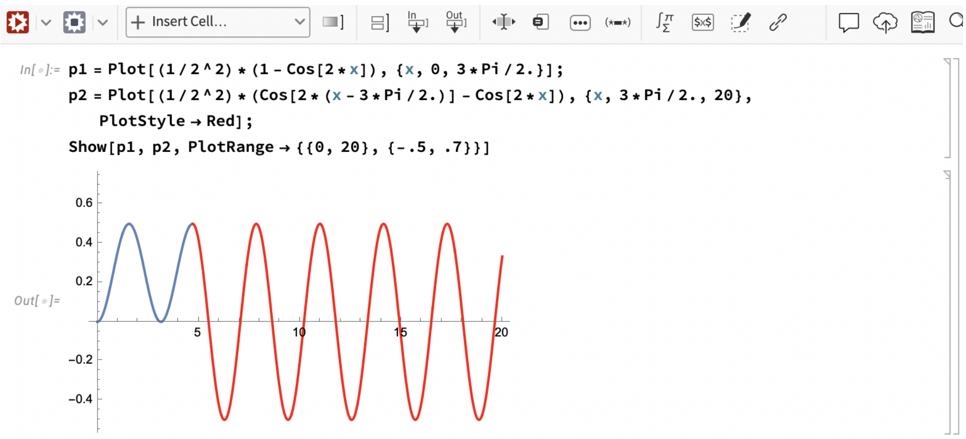
$$\rightarrow \quad Y = \frac{1}{p(p^2 + 2^2)} - \frac{e^{-pa}}{p(p^2 + 2^2)} .$$

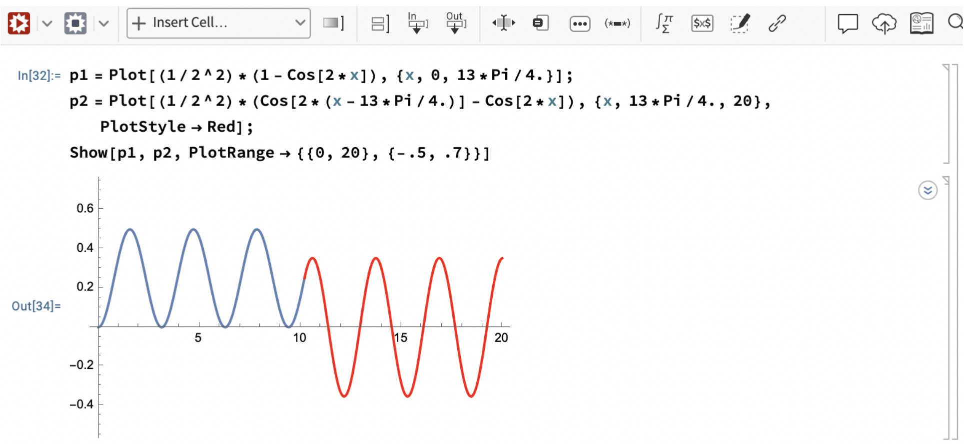
Then, with $L15$ and $L28$ in the Laplace transform table,

$$y(t) = L^{-1}(Y) = \frac{1}{2^2}(1 - \cos 2t) - \frac{1}{2^2}\left[1 - \cos 2(t - a)\right] u(t - a).$$

$$= \begin{cases} \frac{1}{2^2}(1 - \cos 2t), & \text{if } 0 < t < a, \\ \frac{1}{2^2}\left[\cos 2(t - a) - \cos 2t\right] = \frac{1}{2} \sin a \, \sin(2t - a), & \text{if } t > a, \end{cases}$$

where $u(t)$ is the step function (see $L28$ in the Laplace transform table). The resulting response function $y(t)$ to the step function-like external force is illustrated below, with the force switched off *(i)* at $a = \frac{3}{2}\pi$ and *(ii)* at $a = \frac{13}{4}\pi$. In both cases, the period of the oscillation remains as $\frac{2\pi}{2} = \pi$ before and after the force is turned off. In addition, for $t > a$ the oscillation midpoint shifts from 0.25 to 0, while $y(t)$ and $y'(t)$ remain continuous at $t = a$. However, the oscillation amplitude for $t > a$, given by $\frac{1}{2} \sin a$, differs between *(i)* and *(ii)*.





• (b) We first compute the response of this system to a unit impulse at time $t = t'$ (> 0):

$$y'' + 4y = \delta(t - t').$$

With Eqs.(9.1)-(9.2) and (11.7) of Boas Chapter 8, or $L27$ in the Laplace transform table, we transform the differential equation into

$$p^2 Y + 4Y = L[\delta(t - t')] = e^{-pt'} \quad \rightarrow \quad Y = \frac{e^{-pt'}}{p^2 + 2^2}.$$

Then, from $L3$ and $L28$ in the Laplace transform table, you acquire the desired response function (i.e., the Green function) as

$$G(t, t') = L^{-1}(Y) = \begin{cases} 0, & \text{if } t < t', \\ \frac{1}{2}\sin 2(t - t'), & \text{if } t > t', \end{cases}$$

which is essentially the same as Eq.(12.5) of Boas Chapter 8. Using Eq.(12.4) or (12.6), you can write the response of the system to $f(t)$ as

$$\begin{aligned} y(t) &= \int_0^\infty G(t, t') f(t') dt' = \int_0^t \frac{1}{2}\sin 2(t - t') f(t')\, dt' \\ &= \begin{cases} \displaystyle\int_0^t \frac{1}{2}\sin 2(t - t')\, dt' = \left[\frac{1}{2^2}\cos 2(t - t')\right]_0^t = \frac{1}{2^2}(1 - \cos 2t), & \text{if } 0 < t < a, \\ \displaystyle\int_0^a \frac{1}{2}\sin 2(t - t')\, dt' = \left[\frac{1}{2^2}\cos 2(t - t')\right]_0^a = \frac{1}{2^2}\left[\cos 2(t - a) - \cos 2t\right], & \text{if } t > a. \end{cases} \end{aligned}$$

• (c-1) We write the solution in the form of Eq.(4.7) of Boas Chapter 13 as

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \cos\frac{n\pi v t}{l}.$$

At $t = 0$ you want

$$y_0(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \begin{cases} \frac{8hx}{l}, & 0 < x < \frac{l}{8}, \\ 2h - \frac{8hx}{l}, & \frac{l}{8} < x < \frac{l}{4}, \\ 0, & \frac{l}{4} < x < l, \end{cases}$$

from which you can get the Fourier coefficients as

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l y_0(x) \sin\frac{n\pi x}{l} dx \\ &= \frac{2}{l}\int_0^{l/8} \frac{8hx}{l} \sin\frac{n\pi x}{l} dx + \frac{2}{l}\int_{l/8}^{l/4} \left(2h - \frac{8hx}{l}\right) \sin\frac{n\pi x}{l} dx \\ &= \frac{2}{l}\left\{\cancel{\left[-\frac{8hx}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^{l/8}} + \int_0^{l/8} \frac{8h}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l} dx\right\} + \cancel{\left[-\frac{4h}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_{l/8}^{l/4}} \\ &\quad - \frac{2}{l}\left\{\cancel{\left[-\frac{8hx}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_{l/8}^{l/4}} + \int_{l/8}^{l/4} \frac{8h}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l} dx\right\} = \frac{16h}{n^2\pi^2}\left(2\sin\frac{n\pi}{8} - \sin\frac{n\pi}{4}\right). \end{aligned}$$

Therefore,

$$y(x,t) = \frac{16h}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(2\sin\frac{n\pi}{8} - \sin\frac{n\pi}{4}\right)\sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}.$$

• (c-2) Now with the new boundary condition (free end at $x = l$), we write the solution in the form similar to Eq.(4.16) of Boas Chapter 13 as

$$y = \sum_{n=0}^{\infty} b_n \sin\frac{(n+\frac{1}{2})\pi x}{l}\cos\frac{(n+\frac{1}{2})\pi vt}{l}.$$

At $t = 0$ you want

$$y_0(x) = \sum_{n=0}^{\infty} b_n \sin\frac{(n+\frac{1}{2})\pi x}{l} = \begin{cases} \frac{8hx}{l}, & 0 < x < \frac{l}{8}, \\ 2h - \frac{8hx}{l}, & \frac{l}{8} < x < \frac{l}{4}, \\ 0, & \frac{l}{4} < x < l, \end{cases}$$

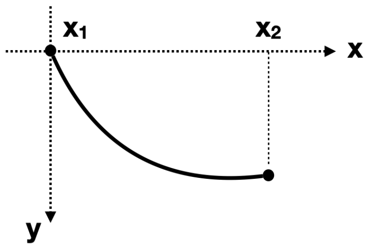
from which you can get the Fourier coefficients as follows (note that the basis functions $\sin\frac{(n+\frac{1}{2})\pi x}{l}$ forms a complete set for our problem; see the last paragraph of Boas Chapter 7, Section 11):

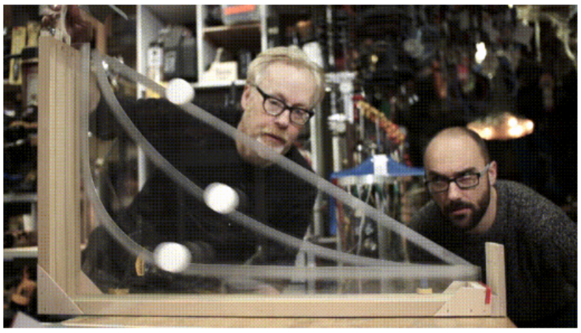
$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l y_0(x)\sin\frac{(n+\frac{1}{2})\pi x}{l}dx \\
&= \frac{2}{l}\int_0^{l/8}\frac{8hx}{l}\sin\frac{(n+\frac{1}{2})\pi x}{l}dx + \frac{2}{l}\int_{l/8}^{l/4}\left(2h - \frac{8hx}{l}\right)\sin\frac{(n+\frac{1}{2})\pi x}{l}dx \\
&= \frac{2}{l}\left\{\cancel{\left[-\frac{8hx}{(n+\frac{1}{2})\pi}\cos\frac{(n+\frac{1}{2})\pi x}{l}\right]_0^{l/8}} + \int_0^{l/8}\frac{8h}{(n+\frac{1}{2})\pi}\cos\frac{(n+\frac{1}{2})\pi x}{l}dx\right\} + \cancel{\left[-\frac{4h}{(n+\frac{1}{2})\pi}\cos\frac{(n+\frac{1}{2})\pi x}{l}\right]_{l/8}^{l/4}} \\
&\quad - \frac{2}{l}\left\{\cancel{\left[-\frac{8hx}{(n+\frac{1}{2})\pi}\cos\frac{(n+\frac{1}{2})\pi x}{l}\right]_{l/8}^{l/4}} + \int_{l/8}^{l/4}\frac{8h}{(n+\frac{1}{2})\pi}\cos\frac{(n+\frac{1}{2})\pi x}{l}dx\right\} \\
&= \frac{64h}{(2n+1)^2\pi^2}\left(2\sin\frac{(2n+1)\pi}{16} - \sin\frac{(2n+1)\pi}{8}\right) \\
&= \frac{64h}{(2n+1)^2\pi^2}\left[2\sin\frac{(2n+1)\pi}{16}\left(1 - \cos\frac{(2n+1)\pi}{16}\right)\right] \\
&= \frac{64h}{(2n+1)^2\pi^2}\left[2\sin\frac{(2n+1)\pi}{16}\cdot 2\sin^2\frac{(2n+1)\pi}{32}\right].
\end{aligned}$$

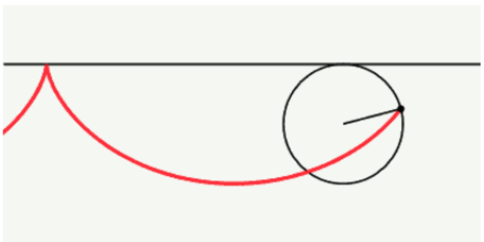
Therefore,

$$\begin{aligned}
y(x,t) &= \frac{64h}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left(2\sin\frac{(2n+1)\pi}{16} - \sin\frac{(2n+1)\pi}{8}\right)\sin\frac{(n+\frac{1}{2})\pi x}{l}\cos\frac{(n+\frac{1}{2})\pi vt}{l} \\
&= \frac{256h}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left(\sin\frac{(2n+1)\pi}{16}\sin^2\frac{(2n+1)\pi}{32}\right)\sin\frac{(n+\frac{1}{2})\pi x}{l}\cos\frac{(n+\frac{1}{2})\pi vt}{l}.
\end{aligned}$$

5. [2 pt] A particle slides downward under gravity along a curve from point $(x_1, y_1) = (0, 0)$ to a lower point (x_2, y_2). Assuming that the particle is released from rest and slides without friction (see figure; note the direction of y-axis), write down the transit time along a path $y(x)$ between the two points as an integral, and solve the Euler equation to make the integral stationary. If needed, change the independent variable to make the equation simpler. You will find that the resulting curve is the well-known cycloid. (Note: The gravitational acceleration is g. To simplify the answer, you may consider parametric form $x(\theta)$ and $y(\theta)$, and first express y as constant multiple of $(1-\cos\theta)$. Once you have $y(\theta)$, obtain $x(\theta)$ by *explicitly* integrating $\frac{dx}{d\theta}$.)







[image credits: youtube.com/watch?v=skvnj67YGmw (*left*), Wikipedia commons (*right*)]

• As described in Boas Chapter 9, Section 4, the time of transit T is written as

$$T = \int_{x_1}^{x_2} \sqrt{\frac{1+y'^2}{2gy}}\, dx.$$

Then, as in Example 3 of Boas Chapter 9, Section 3, with $x' = \frac{dx}{dy} = \frac{1}{y'}$, we change the variable as

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}} dx = \int_{y_1}^{y_2} \frac{\sqrt{1+y'^2}}{\sqrt{y}} x' dy = \int_{y_1}^{y_2} \frac{\sqrt{x'^2+1}}{\sqrt{y}} dy.$$

Now, the Euler equation becomes

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x'} = 0 = \frac{\partial}{\partial y}\left(\frac{x'}{\sqrt{y}\sqrt{x'^2+1}}\right) \;\rightarrow\; \frac{x'^2}{y(x'^2+1)} = C_1 \;\rightarrow\; x' = \frac{\sqrt{C_1 y}}{\sqrt{1-C_1 y}}.$$

However, since this $\frac{dx}{dy}$ is not easily integrable, we instead use parametric representation $x(\theta)$ and $y(\theta)$. By setting $C_1 y = \frac{1}{2}(1-\cos\theta)$,

$$x' = \frac{\sqrt{C_1 y}}{\sqrt{1-C_1 y}} = \sqrt{\frac{\frac{1}{2}(1-\cos\theta)}{\frac{1}{2}(1+\cos\theta)}} = \sqrt{\tan^2\frac{\theta}{2}} = \tan\frac{\theta}{2} = \frac{dx}{dy},$$

from which you get

$$dx = \tan\frac{\theta}{2}dy = \tan\frac{\theta}{2}\cdot\frac{1}{2C_1}\sin\theta\,d\theta = \frac{1}{C_1}\frac{\sin(\theta/2)}{\cos(\theta/2)}\cdot\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\theta = \frac{1}{C_1}\sin^2\frac{\theta}{2}d\theta$$

$$\rightarrow x = \frac{1}{C_1}\int\sin^2\frac{\theta}{2}d\theta = \frac{1}{2C_1}\int(1-\cos\theta)d\theta = \frac{1}{2C_1}(\theta-\sin\theta)+C_2.$$

After setting $C_2 = 0$ using the initial condition ($x = y = 0$ at $\theta = t = 0$), you finally acquire $x(\theta) = \frac{1}{2C_1}(\theta - \sin\theta)$ and $y(\theta) = \frac{1}{2C_1}(1 - \cos\theta)$, consistent with Eq.(4.3) of Boas Chapter 9.

6. (a) [1 pt] Throughout the semester we discussed many examples in which simple mathematical concepts are utilized to understand seemingly complex physical or daily phenomena. In this regard, five of your peers presented their term projects in the last class of the semester (+ more students on eTL). Describe the key idea of one of the presentations you found interesting. A paragraph of at least 3-4 sentences is expected to clearly convey the core idea of his/her term project. If you were one of the presenters, please choose someone else's.

(b) [1 pt] We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques such as order-of-magnitude estimation and/or dimensional analysis. Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem setup. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in $\sim$3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.

- (a) See the student presentation slides in Lecture 15-1 that include the collection of term project presentations by students on December 8.
- (b) See the class slides for Lecture 12-2 that include many example problems, and the grading guideline.