Mathematical Physics I (Fall 2025): Midterm Exam Solution

Oct. 25, 2025

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

1. (a) [1 pt] Find the Maclaurin series expansion for $\arctan x$ by verifying and using the identity:

$$\arctan x = \int_0^x \frac{du}{1+u^2}.$$

This well-known Maclaurin series was first derived by Gregory (1671). Use this result to prove the Leibniz formula for π (1673),[1] that is,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

(b) [2 pt] The special relativistic sum w of two velocities u and v in the same direction is given by

$$\frac{w}{c} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}.$$

Let $\frac{u}{c} = \frac{v}{c} = 1 - \alpha$, with $0 \le \alpha \le 1$. Find the expansion of $\frac{w}{c}$ in powers of α through terms in α^3, which can be useful if $\alpha \ll 1$. (Note: You may want to first express $\frac{w}{c}$ in the form of $\frac{1}{1+f(\alpha)}$.)

• (a) Using the substitution $u = \tan\theta$ and $du = \sec^2\theta$, one can easily verify the given identity. Then, $\arctan x = \int_0^x \frac{du}{1+u^2} = \int_0^x (1 - u^2 + u^4 - u^6 + \dots) du = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$, which yields the Leibniz formula for π when evaluated at $x = 1$.

• (b) $\frac{w}{c} = \frac{(1-\alpha)+(1-\alpha)}{1+(1-\alpha)^2} = \frac{2(1-\alpha)}{2(1-\alpha)+\alpha^2} = \frac{1}{1+\frac{\alpha^2}{2(1-\alpha)}} = \frac{1}{1+f(\alpha)}$. Now, expanding the terms in binomial power series, $\frac{w}{c} = \frac{1}{1+f(\alpha)} = 1 - f(\alpha) + f(\alpha)^2 - \cdots = 1 - \frac{\alpha^2}{2}\{1 + \alpha + \alpha^2 + O(\alpha^3)\} + \left(\frac{\alpha^2}{2}\right)^2 \{1 + \alpha + \alpha^2 + O(\alpha^3)\}^2 - \cdots = 1 - \frac{\alpha^2}{2} - \frac{\alpha^3}{2} + O(\alpha^4)$.

[1] In fact, Leibniz used a completely different, geometrical method to derive this formula. For historical context, see *"The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha"* (R. Roy, 1990).

2. (a) [1 pt] Show that $\sinh z = \sinh x \cos y + i \cosh x \sin y$, where $z = x + iy$ is a complex number.

(b) [2 pt] Evaluate the complex expression $e^{2\tanh^{-1} i}$ in a simple $x + iy$ form.

• (a) $\sinh x \cos y + i \cosh x \sin y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^{iy} + e^{-iy}}{2} + i \cdot \frac{e^x + e^{-x}}{2} \cdot \frac{e^{iy} - e^{-iy}}{2i} = 2 \cdot \frac{e^{x+iy} - e^{-x-iy}}{4} = \frac{e^z - e^{-z}}{2} = \sinh z$.

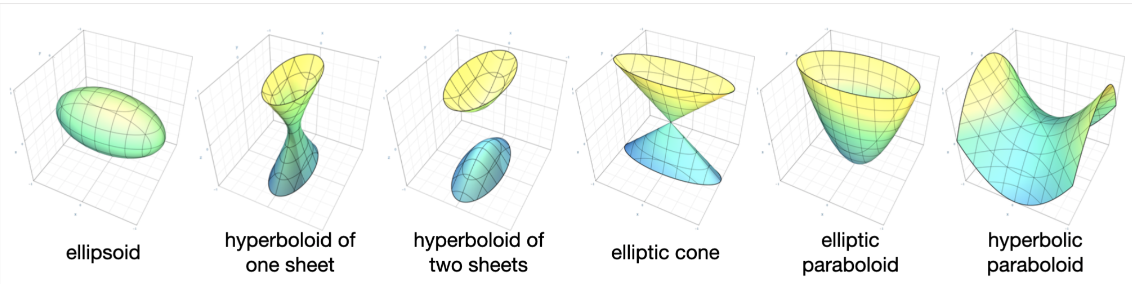
• (b) $\tanh^{-1} i = w \;\rightarrow\;$ With $u \equiv e^\omega$, we find $\tanh w = \frac{e^\omega - e^{-\omega}}{e^\omega + e^{-\omega}} = \frac{u - u^{-1}}{u + u^{-1}} = i \;\rightarrow\; u^2 = \frac{1+i}{1-i} = i$ $\rightarrow$ then, what we wish to acquire is $e^{2\tanh^{-1} i} = e^{2\omega} = u^2 = i$.

3. (a) [2 pt] Solve the set of equations below by <u>*all three*</u> methods listed here: *(i)* by row reducing the augmented matrix, *(ii)* by using Cramer's rule, *(iii)* by finding the inverse of the coefficient matrix. Clearly indicate which method you are using for each part of your answer, so that the grader could follow it easily.

$$\begin{cases} x + 2y - \;\; z = -3 \\ 2x - \;\; y + 4z = \;\; 17 \\ -3x + 5y + 2z = -5 \end{cases}$$

(b) [2 pt] Find the shortest distance from the origin to the line of intersection of the two planes $2x - 3y + z = 5$ and $3x - y - 2z = 11$, by first determining the direction of the line of intersection using vector methods. (Note: Here, you may easily guess — but simply use — the fact that a point $P = (4, 1, 0)$ lies on both planes.)

(c) [2 pt] Find the shortest distance from the origin to the quadric surface $6xy + 2z^2 = 3$, by rotating it to its principal axes (x', y', z'). For your information, examples of quadric surfaces in 3 dimensions include, but are not limited to, the following, as discussed in the class.



• (a-1) As in Example 1 of Boas Chapter 3, Section 2,

$$\begin{pmatrix} 1 & 2 & -1 & -3 \\ 2 & -1 & 4 & 17 \\ -3 & 5 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & -5 & 6 & 23 \\ 0 & 11 & -1 & -14 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & -5 & 6 & 23 \\ 0 & 0 & \frac{61}{5} & \frac{183}{5} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & -\frac{6}{5} & -\frac{23}{5} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

• (a-2) As in Example 5 of Boas Chapter 3, Section 3,

$$x = \begin{vmatrix} -3 & 2 & -1 \\ 17 & -1 & 4 \\ -5 & 5 & 2 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ -3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} -3 & 2 & -1 \\ 5 & 7 & 0 \\ -11 & 9 & 0 \end{vmatrix} \Big/ \begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 6 \\ 0 & 11 & -1 \end{vmatrix} = -\begin{vmatrix} 5 & 7 \\ -11 & 9 \end{vmatrix} \Big/ \begin{vmatrix} -5 & 6 \\ 11 & -1 \end{vmatrix} = 2, \text{ etc.}$$

• (a-3) As in Example 3 of Boas Chapter 3, Section 6,

$$M = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ -3 & 5 & 2 \end{pmatrix} \to \det(M) = -61 \text{ and } C = \begin{pmatrix} -22 & -16 & 7 \\ -9 & -1 & -11 \\ 7 & -6 & -5 \end{pmatrix}$$

$$\to M^{-1} = \frac{1}{\det(M)} C^T = \frac{1}{61} \begin{pmatrix} 22 & 9 & -7 \\ 16 & 1 & 6 \\ -7 & 11 & 5 \end{pmatrix} \to \text{ then, } x = 2, \text{ etc.}$$

• (b) As in Example 6 of Boas Chapter 3, Section 5, the direction of the intersection is the cross product of the two normal vectors, $(2, -3, 1) \times (3, -1, -2) = (7, 7, 7)$, which gives the line of intersection as $\mathbf{r} = 4\mathbf{i} + \mathbf{j} + (\mathbf{i} + \mathbf{j} + \mathbf{k})t$. From Figure 5.7 and the accompanying equations, the distance between O and the line is $|\overrightarrow{OP} \times \mathbf{u}| = \left|(4, 1, 0) \times \frac{1}{\sqrt{3}}(1, 1, 1)\right| = \left|\frac{1}{\sqrt{3}}(1, -4, 3)\right| = \sqrt{\frac{26}{3}}$.

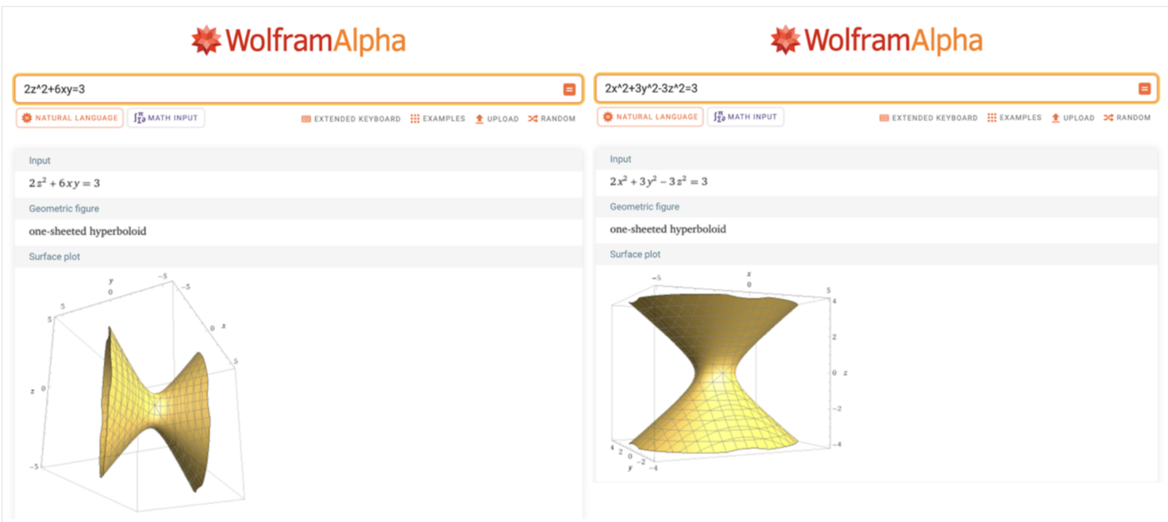
• (c) As in Example 2 of Boas Chapter 3, Section 12,

$$(x, y, z) \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \to \begin{vmatrix} -\lambda & 3 & 0 \\ 3 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 = (2-\lambda)(\lambda+3)(\lambda-3)$$

from which $\lambda = 2$, 3 and -3 are found. Therefore, the new quadric surface equation relative to the principal axes becomes

$$(x', y', z') \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = 3 \to \frac{2x'^2}{3} + y'^2 - z'^2 = 1$$

which represents a hyperboloid of one sheet. At $(x', y', z') = (0, \pm 1, 0)$, the surface has the shortest distance to the origin, $d = 1$.



4. (a) [1 pt] Show that for $x \in \left[0, \frac{\pi}{2}\right]$,

$$\frac{d}{dx}\int_{\sin x}^{\cos x} \sqrt{1-t^2}dt = -1.$$

(b) [1 pt] Repeat Problem 3(b), this time using the method of Lagrange multipliers. (Note: This problem may be somewhat algebraically intensive; please be advised in advance.)

(c) [1 pt] Repeat Problem 3(c), this time using the method of Lagrange multipliers, without carrying out any coordinate transformation.

• (a) Using Eq.(12.8) of Boas Chapter 4,

$$\frac{d}{dx}\int_{\sin x}^{\cos x} \sqrt{1-t^2}dt = \sqrt{1-\cos^2 x}\cdot(-\sin x) - \sqrt{1-\sin^2 x}\cdot(\cos x) = -1.$$

• (b) From Thm.(9.20) of Boas Chapter 4, and following Example 4 of Boas Chapter 4, Section 9, we write $F = x^2+y^2+z^2+\lambda_1(2x-3y+z)+\lambda_2(3x-y-2z)$. Then, $\frac{\partial F}{\partial x} = 2x+2\lambda_1+3\lambda_2 = 0$, $\frac{\partial F}{\partial y} = 2y-3\lambda_1-\lambda_2 = 0$, and $\frac{\partial F}{\partial z} = 2z+\lambda_1-2\lambda_2 = 0$. Combining the equations with the constraints $2x-3y+z=5$ and $3x-y-2z=11$ gives $(x,y,z) = \frac{1}{3}(7,-2,-5)$ with $\lambda_1 = \frac{2}{21}$ and $\lambda_2 = -\frac{34}{21}$. Thus, $d = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{26}{3}}$.

• (c) From Eq.(9.6) and Thm.(9.20) of Boas Chapter 4, and following Example 3 of Boas Chapter 4, Section 10, we write $F = x^2+y^2+z^2+\lambda(6xy+2z^2)$. Then, $\frac{\partial F}{\partial x} = 2x+6\lambda y = 0$, $\frac{\partial F}{\partial y} = 2y+6\lambda x = 0$, and $\frac{\partial F}{\partial z} = 2z+4\lambda z = 0$. The last equation gives two possibilities of extremum conditions:

- (i) $\lambda = -\frac{1}{2}$ $\rightarrow$ $2x-3y=0$ and $2y-3x=0$ $\rightarrow$ $x=y=0$ $\rightarrow$ the constraint $6xy+2z^2=3$ then gives $(x,y,z) = (0,0,\pm\sqrt{\frac{3}{2}})$ and the distance to the origin, $d = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{3}{2}}$.

- (ii) $z=0$ $\rightarrow$ for $2x+6\lambda y=0$ and $6\lambda x+2y=0$ to have a nontrivial solution, we should have $\lambda = \pm\frac{1}{3}$ (acquired from the determinant) $\rightarrow$ $y=\pm x$ $\rightarrow$ putting everything into the constraint $6xy+2z^2=3$ gives $(x,y,z) = (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0)$ and the distance to the origin, $d=1$, which is obviously smaller than what we got in (i).

5. [2 pt] Compute the gravitational force on a unit mass located at the origin, due to the mass of uniform density ρ occupying the volume inside the sphere $r=2a$ (centered at the origin) and above the plane $z=a$. (Note: Make sure to specify both the magnitude and the direction of the resulting force. You may invoke symmetry arguments to explain why certain components of the net force are zero. The gravitational constant is G.)

• In spherical coordinates, the magnitude of the gravitational force on the unit mass due to the element of mass dM at (r,θ,ϕ) is $dF = \frac{GdM}{r^2} = \frac{G\rho dV}{r^2}$. So, the z-component of the force is

$$F_z = \int dF\cos\theta = \int \frac{G\rho}{r^2}\cdot r^2\sin\theta\, dr d\theta d\phi\cdot\cos\theta = G\rho\int_0^{2\pi} d\phi \int_0^{\frac{\pi}{3}} \sin\theta\cos\theta\, d\theta \int_{\frac{a}{\cos\theta}}^{2a} dr$$

$$= 2\pi a G\rho \int_0^{\frac{\pi}{3}} \sin\theta\cos\theta\, d\theta \left[2 - \frac{1}{\cos\theta}\right] = 2\pi a G\rho\int_0^{\frac{\pi}{3}} \left[\sin 2\theta - \sin\theta\right] d\theta = \frac{1}{2}\pi a G\rho,$$

where we adjusted the range of r as a function of θ, while keeping θ's range at $[0, \frac{\pi}{3}]$. Alternatively, you may adjust the range of θ as a function of r, while keeping r's range at $[a, 2a]$:

$$\begin{aligned}
F_z &= \int dF \cos\theta = \int \frac{G\rho}{r^2} \cdot r^2 \sin\theta \, dr d\theta d\phi \cdot \cos\theta = G\rho \int_0^{2\pi} d\phi \int_a^{2a} dr \int_0^{\arccos\left(\frac{a}{r}\right)} \sin\theta\cos\theta d\theta \\
&= 2\pi G\rho \int_a^{2a} dr \left[-\frac{1}{2}\cos 2\theta\right]_0^{\arccos\left(\frac{a}{r}\right)} = 2\pi G\rho \int_a^{2a} dr \left[\frac{1}{2} - \cos^2\theta\right]_0^{\arccos\left(\frac{a}{r}\right)} \\
&= 2\pi G\rho \int_a^{2a} dr \left[1 - \frac{a^2}{r^2}\right] = 2\pi G\rho \left[r + \frac{a^2}{r}\right]_a^{2a} = \frac{1}{2}\pi a G\rho.
\end{aligned}$$

Either way, you find that the force exerted by the "spherical cap" on the unit mass at the origin is $\mathbf{F} = F_z\hat{\mathbf{z}} = \frac{1}{2}\pi a G\rho\hat{\mathbf{z}}$.

6. (a) [2 pt] Evaluate the integral

$$\iint_\sigma (\nabla \times \mathbf{V}) \cdot \mathbf{n} d\sigma$$

over the surface σ consisting of the four slanting faces of a pyramid whose base is the square in the (x, y)-plane with corners at $(0, 0)$, $(0, 2)$, $(2, 0)$, $(2, 2)$ and whose top vertex is at $(1, 1, 2)$, where $\mathbf{V} = (x^2 z - 2)\mathbf{i} + (x + y - z)\mathbf{j} - xyz\mathbf{k}$. Here, $\mathbf{n}$ denotes the unit normal vector to σ, pointing outward from the pyramid.

(b) [1 pt] Prove that $\nabla \times (\phi\mathbf{V}) = \phi(\nabla \times \mathbf{V}) - \mathbf{V} \times (\nabla\phi)$. Here you are asked to prove the identity by explicitly working with the components, i.e.,

$$[\nabla \times (\phi\mathbf{V})]_x = \frac{\partial(\phi V_z)}{\partial y} - \frac{\partial(\phi V_y)}{\partial z} = ...$$

You are also welcomed to tackle this problem using tensor notation for an additional +1 point.

- (a-1) Using Stokes' theorem, Eq.(11.9) of Boas Chapter 6,

$$\begin{aligned}
\iint_\sigma (\nabla \times \mathbf{V}) \cdot \mathbf{n} d\sigma &= \oint_{\partial\sigma} \mathbf{V} \cdot d\mathbf{r} \\
&= \int_0^2 (-2)\mathbf{i} \cdot \mathbf{i} dx + \int_0^2 (2 + y)\mathbf{j} \cdot \mathbf{j} dy + \int_2^0 (-2)\mathbf{i} \cdot \mathbf{i} dx + \int_2^0 y\,\mathbf{j} \cdot \mathbf{j} dy \\
&= \int_0^2 (2 + y) dy - \int_0^2 y dy = \int_0^2 2\, dy = 4.
\end{aligned}$$

- (a-2) Alternatively, you can carry out a different surface integral over σ', the "square" in the (x, y)-plane as defined in the problem:

$$\begin{aligned}
\iint_\sigma (\nabla \times \mathbf{V}) \cdot \mathbf{n} d\sigma &= \iint_{\sigma'} (\nabla \times \mathbf{V}) \cdot \mathbf{k} d\sigma' = \int_0^2 \int_0^2 \left[(-xz + 1)\mathbf{i} + (x^2 + yz)\mathbf{j} + \mathbf{k}\right] \cdot \mathbf{k} dx dy \\
&= \int_0^2 \int_0^2 dx dy = 4.
\end{aligned}$$

• (b-1) Working with the vector components,

$$
\begin{aligned}
[\nabla \times (\phi \mathbf{V})]_x &= \frac{\partial(\phi V_z)}{\partial y} - \frac{\partial(\phi V_y)}{\partial z} = \left(\frac{\partial \phi}{\partial y} V_z + \phi \frac{\partial V_z}{\partial y} \right) - \left(\frac{\partial \phi}{\partial z} V_y + \phi \frac{\partial V_y}{\partial z} \right) \\
&= \phi \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) - [V_y (\nabla \phi)_z - V_z (\nabla \phi)_y] \\
&= \phi (\nabla \times \mathbf{V})_x - [\mathbf{V} \times (\nabla \phi)]_x \, .
\end{aligned}
$$

• (b-2) Alternatively, writing in tensor notation and then using Eqs. (5.11), (5.13) and others of Boas Chapter 10,

$$
\begin{aligned}
[\nabla \times (\phi \mathbf{V})]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\phi \mathbf{V})_k = \epsilon_{ijk} \left(\frac{\partial \phi}{\partial x_j} V_k \right) + \epsilon_{ijk} \left(\phi \frac{\partial V_k}{\partial x_j} \right) \\
&= \phi \cdot \epsilon_{ijk} \left(\frac{\partial V_k}{\partial x_j} \right) - \epsilon_{ikj} V_k (\nabla \phi)_j \\
&= \phi (\nabla \times \mathbf{V})_i - [\mathbf{V} \times (\nabla \phi)]_i \, .
\end{aligned}
$$