Mathematical Physics I (Fall 2025): Final Examination

Dec. 13, 2025

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

• First, make sure you have all 6 answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from ① to ⑥ on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet ②). After the exam, you will separately turn in all 6 answer sheets, even if some sheets are still blank.

• Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you find any issue or question, you *must* raise it in the first 30 minutes. You have to stay in the room for that 30 minutes even if you have nothing to write down.

• Make your writing easy to read. Illegible answers will *not* be graded.

1. (a) [2 pt] Find the exponential Fourier transform and the sine transform of

$$f(x) = \begin{cases} 1, & -2 < x < 0, \\ -1, & 0 < x < 2, \\ 0 & \text{otherwise}, \end{cases}$$

and use your result to evaluate

$$\int_0^\infty \frac{\sin^3\alpha}{\alpha}\, d\alpha\,.$$

(Note: For the latter question, you may want to recall the trigonometric identity $\frac{1-\cos 2y}{2} = \sin^2 y$.)

(b) [2 pt] The periodic function $f(x) = |x|$ is given over one period $-\frac{\pi}{2} < x < \frac{\pi}{2}$. Sketch several periods of $f(x)$. Expand $f(x)$ in an appropriate Fourier series, and use your result to evaluate

$$\sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \ldots$$

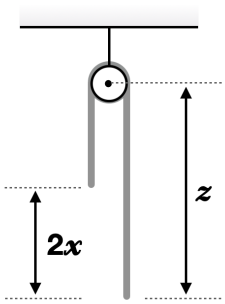
2. Find the general solution to each of the following differential equations.

(a) [2 pt] $(D^2 + 1)y = 8x \sin x + x^2 - x$

(b) [1 pt] $(y + 2x)dx - xdy = 0$

3. (a) [2 pt] A flexible chain of total length l hangs over a frictionless peg of negligible size, with one end of the chain slightly longer than the other (see figure). Let $2x$ denote the difference in length between the two ends of the chain and take $x = x_0$ when $t = 0$. Assuming that the chain is released from rest and slides off without friction, write down the differential equation of motion using the variable x. Show that, for $0 < x < \frac{l}{2}$,

$$x(t) = x_0 \cosh\left(t\sqrt{\frac{2g}{l}}\right).$$



(b) [2 pt] Show that the Lagrangian of a particle of rest mass m in a motion along x-axis,

$$L = mc^2(1 - \sqrt{1 - v^2/c^2}) - V(x),$$

leads to a relativistic equation of motion,

$$F = \frac{d}{dt}\frac{mv}{\sqrt{1 - v^2/c^2}}.$$

Here, c is the speed of light, t and $v = \frac{dx}{dt}$ are the ordinary time and velocity, respectively. Then, consider a particle subject to a constant force F. By integrating the separable equation, determine $v(t)$. Verify that $v \to c$ as $t \to \infty$. Finally, compute the distance $s(t)$ traveled by the particle starting from rest. (Note: m is constant. If you encounter an integral of the form $\int \frac{dy}{(1-y^2)^{3/2}}$, consider the substitution $y = \sin\theta$.)

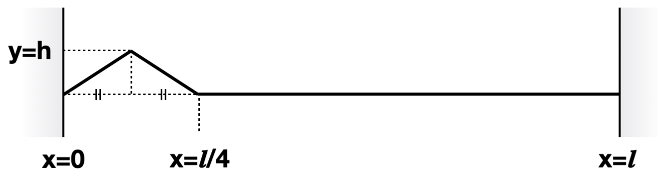
4. (a) [2 pt] A mechanical system is described by the differential equation $y'' + 4y = f(t)$ with

$$f(t) = \begin{cases} 1, & \text{if } 0 < t < a, \\ 0, & \text{if } t > a. \end{cases}$$

Assuming $y_0 = y_0' = 0$ at $t = 0$ and using the Laplace transform, find the response $y(t)$ for $t > 0$. Sketch the motion of the system for two cases: *(i)* $a = \frac{3}{2}\pi$ and *(ii)* $a = \frac{13}{4}\pi$. (Note: The table of Laplace transforms is in the last page of this exam.)

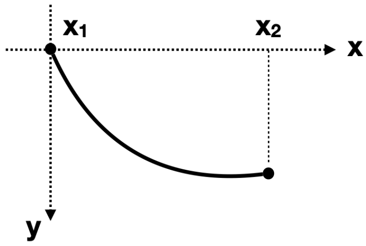
(b) [1 pt] Repeat (a), this time using the Green function. (Note: To utilize the method of Green function, you will first need to explicitly derive the response of this system to a unit impulse at time $t = t'\,(> 0)$. Again, the table of Laplace transforms is in the last page of this exam.)

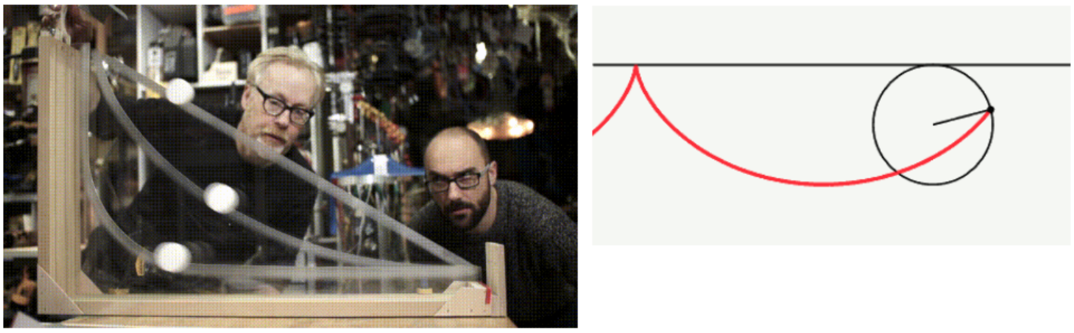
(c) [2 pt] A string of length l is initially at rest and given the triangular displacement y_0 shown below. Assuming the string is fixed at both ends, determine the displacement y as a function of position x, time t, and the wave velocity v (a constant determined by the tension and the linear density of the string). For an additional +1 point, repeat the problem under the new boundary conditions where the string is pinned at $x = 0$ but is free to move up and down at $x = l$.



[Problems 5 and 6 in the next page.]

5. [2 pt] A particle slides downward under gravity along a curve from point $(x_1, y_1) = (0, 0)$ to a lower point (x_2, y_2). Assuming that the particle is released from rest and slides without friction (see figure; note the direction of y-axis), write down the transit time along a path $y(x)$ between the two points as an integral, and solve the Euler equation to make the integral stationary. If needed, change the independent variable to make the equation simpler. You will find that the resulting curve is the well-known cycloid. (Note: The gravitational acceleration is g. To simplify the answer, you may consider parametric form $x(\theta)$ and $y(\theta)$, and first express y as constant multiple of $(1-\cos\theta)$. Once you have $y(\theta)$, obtain $x(\theta)$ by *explicitly* integrating $\frac{dx}{d\theta}$.)





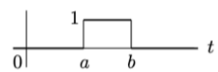
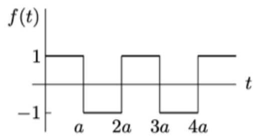
[image credits: youtube.com/watch?v=skvnj67YGmw (*left*), Wikipedia commons (*right*)]

6. (a) [1 pt] Throughout the semester we discussed many examples in which simple mathematical concepts are utilized to understand seemingly complex physical or daily phenomena. In this regard, five of your peers presented their term projects in the last class of the semester (+ more students on eTL). Describe the key idea of one of the presentations you found interesting. A paragraph of at least 3-4 sentences is expected to clearly convey the core idea of his/her term project. If you were one of the presenters, please choose someone else's.

(b) [1 pt] We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques such as order-of-magnitude estimation and/or dimensional analysis. Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem setup. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in ∼3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.

Table of Laplace Transforms

	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt$	
L1	1	$\frac{1}{p}$	Re $p > 0$
L2	e^{-at}	$\frac{1}{p+a}$	Re $(p+a) > 0$
L3	$\sin at$	$\frac{a}{p^2+a^2}$	Re $p > \lvert \text{Im } a \rvert$
L4	$\cos at$	$\frac{p}{p^2+a^2}$	Re $p > \lvert \text{Im } a \rvert$
L5	$t^k,\ k > -1$	$\frac{k!}{p^{k+1}}$ or $\frac{\Gamma(k+1)}{p^{k+1}}$	Re $p > 0$
L6	$t^k e^{-at},\ k > -1$	$\frac{k!}{(p+a)^{k+1}}$ or $\frac{\Gamma(k+1)}{(p+a)^{k+1}}$	Re $(p+a) > 0$
L7	$\frac{e^{-at} - e^{-bt}}{b-a}$	$\frac{1}{(p+a)(p+b)}$	Re $(p+a) > 0$ Re $(p+b) > 0$
L8	$\frac{ae^{-at} - be^{-bt}}{a-b}$	$\frac{p}{(p+a)(p+b)}$	Re $(p+a) > 0$ Re $(p+b) > 0$
L9	$\sinh at$	$\frac{a}{p^2-a^2}$	Re $p > \lvert \text{Re } a \rvert$
L10	$\cosh at$	$\frac{p}{p^2-a^2}$	Re $p > \lvert \text{Re } a \rvert$
L11	$t \sin at$	$\frac{2ap}{(p^2+a^2)^2}$	Re $p > \lvert \text{Im } a \rvert$
L12	$t \cos at$	$\frac{p^2-a^2}{(p^2+a^2)^2}$	Re $p > \lvert \text{Im } a \rvert$
L13	$e^{-at} \sin bt$	$\frac{b}{(p+a)^2+b^2}$	Re $(p+a) > \lvert \text{Im } b \rvert$
L14	$e^{-at} \cos bt$	$\frac{p+a}{(p+a)^2+b^2}$	Re $(p+a) > \lvert \text{Im } b \rvert$
L15	$1 - \cos at$	$\frac{a^2}{p(p^2+a^2)}$	Re $p > \lvert \text{Im } a \rvert$
L16	$at - \sin at$	$\frac{a^3}{p^2(p^2+a^2)}$	Re $p > \lvert \text{Im } a \rvert$
L17	$\sin at - at \cos at$	$\frac{2a^3}{(p^2+a^2)^2}$	Re $p > \lvert \text{Im } a \rvert$

	$y = f(t),\ t > 0$ $[y = f(t) = 0,\ t < 0]$	$Y = L(y) = F(p) = \int_0^\infty e^{-pt} f(t)\,dt$	
L18	$e^{-at}(1 - at)$	$\frac{p}{(p+a)^2}$	Re $(p+a) > 0$
L19	$\frac{\sin at}{t}$	$\arctan \frac{a}{p}$	Re $p > \lvert \text{Im } a \rvert$
L20	$\frac{1}{t} \sin at \cos bt,$ $a > 0,\ b > 0$	$\frac{1}{2}\left(\arctan \frac{a+b}{p} + \arctan \frac{a-b}{p}\right)$	Re $p > 0$
L21	$\frac{e^{-at} - e^{-bt}}{t}$	$\ln \frac{p+b}{p+a}$	Re $(p+a) > 0$ Re $(p+b) > 0$
L22	$1 - \text{erf}\left(\frac{a}{2\sqrt{t}}\right),\quad a > 0$ (See Chapter 11, Section 9)	$\frac{1}{p} e^{-a\sqrt{p}}$	Re $p > 0$
L23	$J_0(at)$ (See Chapter 12, Section 12)	$(p^2+a^2)^{-1/2}$	Re $p > \lvert \text{Im}\, a \rvert$; or Re $p \ge 0$ for real $a \ne 0$
L24	$u(t-a) = \begin{cases} 1, & t > a > 0 \\ 0, & t < a \end{cases}$ (unit step, or Heaviside function)	$\frac{1}{p} e^{-pa}$	Re $p > 0$
L25	$f(t) = u(t-a) - u(t-b)$ 	$\frac{e^{-ap} - e^{-bp}}{p}$	All p
L26		$\frac{1}{p} \tanh\left(\frac{1}{2}ap\right)$	Re $p > 0$
L27	$\delta(t-a),\ a \ge 0$ (See Section 11)	e^{-pa}	
L28	$f(t) = \begin{cases} g(t-a), & t > a > 0 \\ 0, & t < a \end{cases}$ $= g(t-a)u(t-a)$	$e^{-pa}G(p)$ [$G(p)$ means $L(g)$.]	
L29	$e^{-at} g(t)$	$G(p+a)$	