

Mathematical Physics I (Fall 2025): Midterm Examination

Oct. 25, 2025

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

- First, make sure you have all 6 answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from ① to ⑥ on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet ②). After the exam, you will separately turn in all 6 answer sheets, even if some sheets are still blank.
- Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you find any issue or question, you *must* raise it in the first 30 minutes. You have to stay in the room for that 30 minutes even if you have nothing to write down.
- Make your writing easy to read. Illegible answers will *not* be graded.

1. (a) [1 pt] Find the Maclaurin series expansion for $\arctan x$ by verifying and using the identity:

$$\arctan x = \int_0^x \frac{du}{1+u^2}.$$

This well-known Maclaurin series was first derived by Gregory (1671). Use this result to prove the Leibniz formula for π (1673),¹ that is,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

- (b) [2 pt] The special relativistic sum w of two velocities u and v in the same direction is given by

$$\frac{w}{c} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}}.$$

Let $\frac{u}{c} = \frac{v}{c} = 1 - \alpha$, with $0 \leq \alpha \leq 1$. Find the expansion of $\frac{w}{c}$ in powers of α through terms in α^3 , which can be useful if $\alpha \ll 1$. (Note: You may want to first express $\frac{w}{c}$ in the form of $\frac{1}{1+f(\alpha)}$.)

¹In fact, Leibniz used a completely different, geometrical method to derive this formula. For historical context, see “*The Discovery of the Series Formula for π by Leibniz, Gregory and Nilakantha*” (R. Roy, 1990).

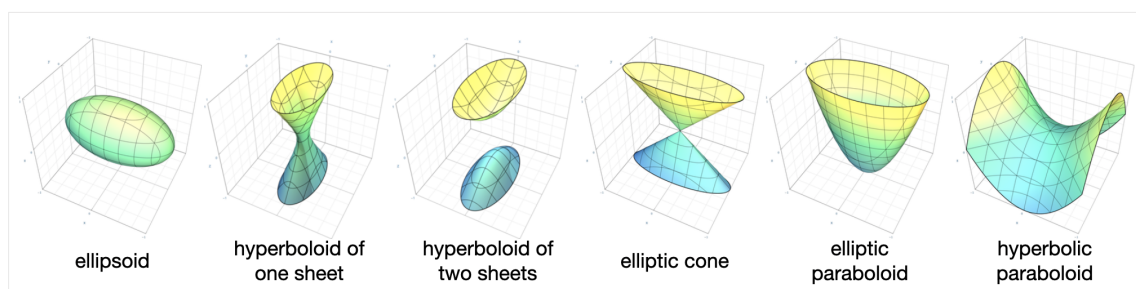
2. (a) [1 pt] Show that $\sinh z = \sinh x \cos y + i \cosh x \sin y$, where $z = x + iy$ is a complex number.
- (b) [2 pt] Evaluate the complex expression $e^{2 \tanh^{-1} i}$ in a simple $x + iy$ form.

3. (a) [2 pt] Solve the set of equations below by all three methods listed here: (i) by row reducing the augmented matrix, (ii) by using Cramer's rule, (iii) by finding the inverse of the coefficient matrix. Clearly indicate which method you are using for each part of your answer, so that the grader could follow it easily.

$$\begin{cases} x + 2y - z = -3 \\ 2x - y + 4z = 17 \\ -3x + 5y + 2z = -5 \end{cases}$$

- (b) [2 pt] Find the shortest distance from the origin to the line of intersection of the two planes $2x - 3y + z = 5$ and $3x - y - 2z = 11$, by first determining the direction of the line of intersection using vector methods. (Note: Here, you may easily guess — but simply use — the fact that a point $P = (4, 1, 0)$ lies on both planes.)

- (c) [2 pt] Find the shortest distance from the origin to the quadric surface $6xy + 2z^2 = 3$, by rotating it to its principal axes (x', y', z') . For your information, examples of quadric surfaces in 3 dimensions include, but are not limited to, the following, as discussed in the class.



4. (a) [1 pt] Show that for $x \in [0, \frac{\pi}{2}]$,

$$\frac{d}{dx} \int_{\sin x}^{\cos x} \sqrt{1-t^2} dt = -1.$$

(b) [1 pt] Repeat Problem 3(b), this time using the method of Lagrange multipliers. (Note: This problem may be somewhat algebraically intensive; please be advised in advance.)

(c) [1 pt] Repeat Problem 3(c), this time using the method of Lagrange multipliers, without carrying out any coordinate transformation.

5. [2 pt] Compute the gravitational force on a unit mass located at the origin, due to the mass of uniform density ρ occupying the volume inside the sphere $r = 2a$ (centered at the origin) and above the plane $z = a$. (Note: Make sure to specify both the magnitude and the direction of the resulting force. You may invoke symmetry arguments to explain why certain components of the net force are zero. The gravitational constant is G .)

6. (a) [2 pt] Evaluate the integral

$$\iint_{\sigma} (\nabla \times \mathbf{V}) \cdot \mathbf{n} d\sigma$$

over the surface σ consisting of the four slanting faces of a pyramid whose base is the square in the (x, y) -plane with corners at $(0, 0)$, $(0, 2)$, $(2, 0)$, $(2, 2)$ and whose top vertex is at $(1, 1, 2)$, where $\mathbf{V} = (x^2z - 2)\mathbf{i} + (x + y - z)\mathbf{j} - xyz\mathbf{k}$. Here, $\mathbf{n}$ denotes the unit normal vector to σ , pointing outward from the pyramid.

(b) [1 pt] Prove that $\nabla \times (\phi \mathbf{V}) = \phi(\nabla \times \mathbf{V}) - \mathbf{V} \times (\nabla \phi)$. Here you are asked to prove the identity by explicitly working with the components, i.e.,

$$[\nabla \times (\phi \mathbf{V})]_x = \frac{\partial(\phi V_z)}{\partial y} - \frac{\partial(\phi V_y)}{\partial z} = \dots$$

You are also welcomed to tackle this problem using tensor notation for an additional +1 point.