Mathematical Physics I (Fall 2022): Homework #6 Solution

Due Dec. 2, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

1. Boas Chapter 8, Problem 11.7

(Note: For Problem 11.7, you will first want to review Problem 11.6(b) and (c), and Example 10.1. You may assume $y_0 = y_0' = 0$ as in Problem 11.6(a). You may opt to utilize the *translation theorems* of the Laplace transform described in Problems 8.19 to 8.27 of Boas Chapter 8.)

• With Eqs.(9.1)-(9.2) and (11.7) or $L27$ in the Laplace transform table (Boas p.469-471), we transform the given differential equation into

$$p^2Y + 2pY + Y = L[\delta(t-t_0)] = e^{-pt_0} \quad \rightarrow \quad Y = \frac{e^{-pt_0}}{(p+1)^2},$$

from which you acquire $y = L^{-1}(Y) = (t-t_0)e^{-(t-t_0)}$ for $t > t_0$, by using $L6$ and $L28$ in the Laplace transform table.

2. Boas Chapter 8, Problem 12.2

(Note: For Problem 12.2, note that we assume $t > 0$ and $y_0 = y_0' = 0$ at $t = 0$; see the text right before Eq.(12.6) in Boas Chapter 8.)

• The response of a system — which can be described as Eq.(12.1) — to a unit impulse at t' is the Green's function $G(t, t')$ in Eq.(12.5). Then from Eq.(12.4) or (12.6), you can write

$$\begin{aligned} y(t) &= \int_0^\infty G(t,t')f(t')dt' = \int_0^t \frac{1}{\omega}\sin\omega(t-t')\sin\omega t'\,dt' \\ &= \frac{1}{2\omega}\int_0^t \left[\cos\omega(t-2t') - \cos\omega t\right]dt' = -\frac{1}{4\omega}\int_t^{-t}\cos\omega u\,du - \frac{t}{2\omega}\cos\omega t = \frac{\sin\omega t - \omega t\cos\omega t}{2\omega^2}. \end{aligned}$$

3. Boas Chapter 13, Problem 2.3

(Note: For Problem 2.3 to 4.2, carefully read the instruction at the beginning of the Problems; that is, you are asked to make a computer plot using the first several terms of your answer.)

• For a semi-finite plate of width π with its long sides and the far end held at $T = 0°$, we can write the solution in the form similar to Eq.(2.9) as

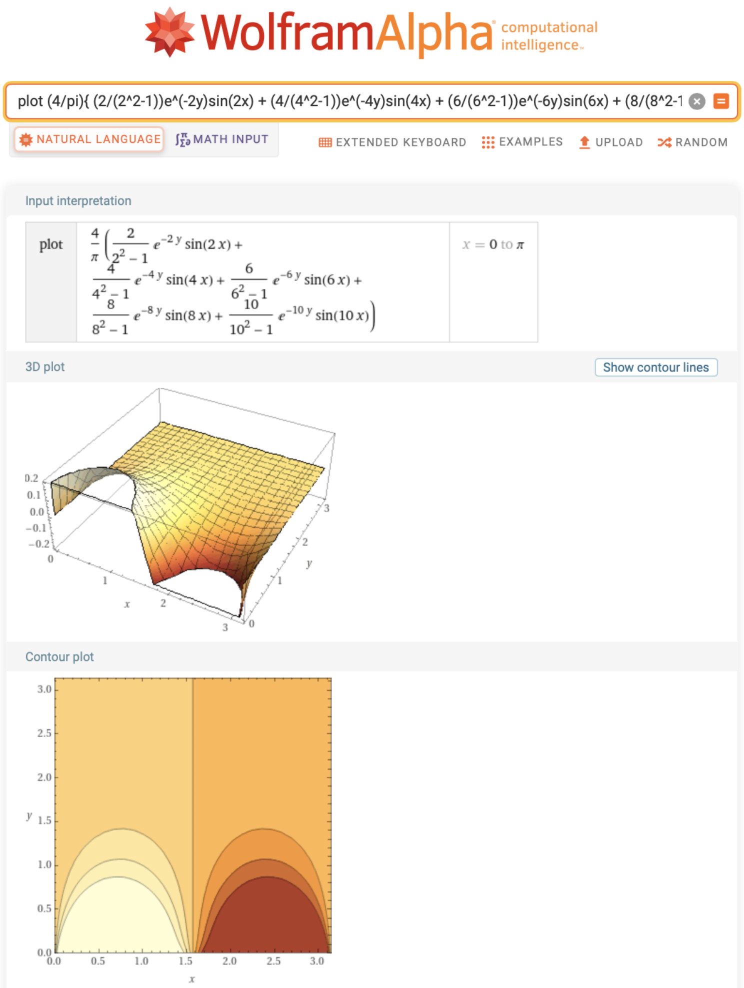
$$T = \sum_{n=1}^{\infty} b_n\, e^{-ny}\sin nx.$$

Because $T_{y=0} = \cos x = \sum_{n=1}^{\infty} b_n \sin nx$, you can find the coefficients of the Fourier sine series (similar to Eq.(9.4) of Boas Chapter 7 if $T_{y=0}$ is treated as an odd function outside $[0, \pi]$) as

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} [\sin(n+1)x + \sin(n-1)x] \, dx = \begin{cases} \frac{4}{\pi}\frac{n}{n^2-1}, & \text{if } n \text{ even}, \\ 0, & \text{if } n \text{ odd}. \end{cases}$$

Therefore,

$$T(x, y) = \sum_{\text{even } n} \frac{4}{\pi}\frac{n}{n^2-1} e^{-ny} \sin nx.$$



4. Boas Chapter 13, Problem 3.3

(Note: For Problem 3.3, you may want to review Example 2 of Boas Chapter 13, Section 3.)

- For the nonzero boundary condition, we can write the solution in the form of Eq.(3.16) as

$$u = 100 - \frac{100x}{l} + \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/l)^2 t} \sin\frac{n\pi x}{l}.$$

At $t = 0$ you want

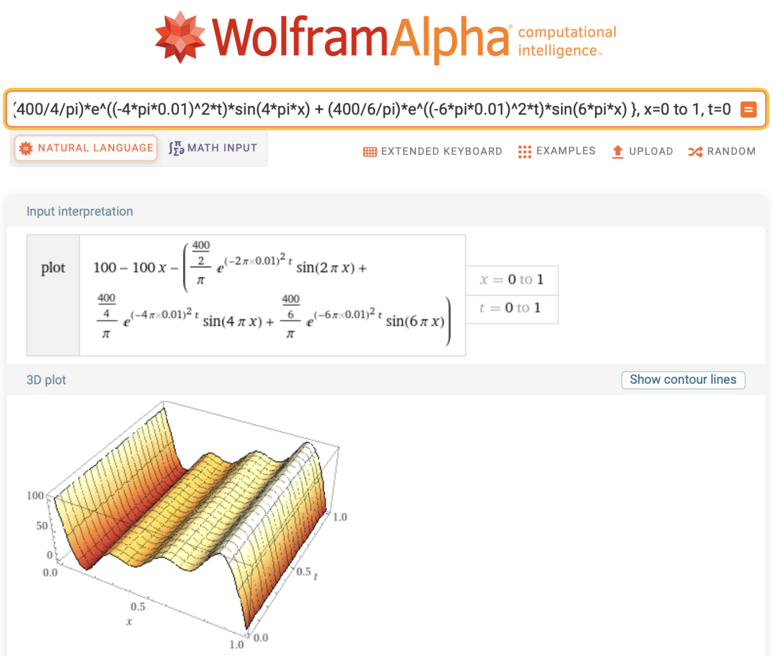
$$u_{t=0} = 100 - \frac{100x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = u_0 = \frac{100x}{l},$$

from which you can get the Fourier coefficients as

$$\begin{aligned}
b_n &= \frac{2}{l}\int_0^l \left(\frac{200x}{l} - 100\right)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l \frac{200x}{l}\sin\frac{n\pi x}{l}dx - \frac{200}{l}\int_0^l \sin\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\left\{\left[-\frac{200x}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l + \int_0^l \frac{200}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}dx\right\} - \left[-\frac{200}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l \\
&= \frac{400}{n\pi}(-1)^{n-1} + \frac{200}{n\pi}((-1)^n - 1) = -\frac{200}{n\pi}(1 + (-1)^n) = \begin{cases} -\frac{400}{n\pi}, & \text{if } n \text{ even,} \\ 0, & \text{if } n \text{ odd.} \end{cases}
\end{aligned}$$

Therefore,

$$u(x,t) = 100 - \frac{100x}{l} + \sum_{\text{even } n}\left(-\frac{400}{n\pi}\right)e^{-(n\pi\alpha/l)^2 t}\sin\frac{n\pi x}{l}.$$



5. Boas Chapter 13, Problem 4.2

- We write the solution in the form of Eq.(4.7) as

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}.$$

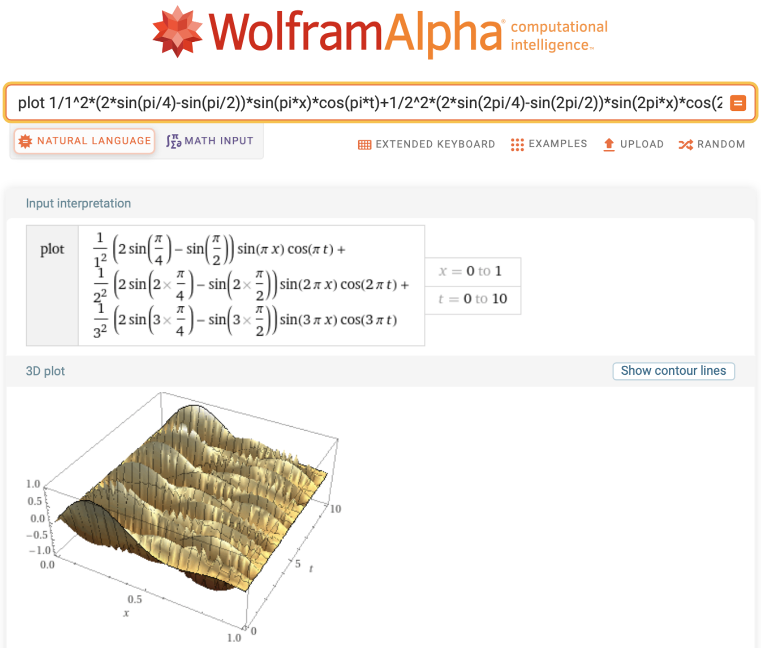
At $t = 0$ you want

$$y_0(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \begin{cases} \frac{4hx}{l}, & 0 < x < \frac{l}{4}, \\ 2h - \frac{4hx}{l}, & \frac{l}{4} < x < \frac{l}{2}, \\ 0, & \frac{l}{2} < x < l, \end{cases}$$

from which you can get the Fourier coefficients as

$$
\begin{aligned}
b_n &= \frac{2}{l}\int_0^l y_0(x)\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\int_0^{l/4}\frac{4hx}{l}\sin\frac{n\pi x}{l}dx + \frac{2}{l}\int_{l/4}^{l/2}\left(2h-\frac{4hx}{l}\right)\sin\frac{n\pi x}{l}dx \\
&= \frac{2}{l}\left\{\left[-\frac{4hx}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^{l/4} + \int_0^{l/4}\frac{4h}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}dx\right\} + \left[-\frac{4h}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_{l/4}^{l/2} \\
&\quad - \frac{2}{l}\left\{\left[-\frac{4hx}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_{l/4}^{l/2} - \int_{l/4}^{l/2}\frac{4h}{l}\frac{l}{n\pi}\cos\frac{n\pi x}{l}dx\right\} = \frac{8h}{n^2\pi^2}\left(2\sin\frac{n\pi}{4}-\sin\frac{n\pi}{2}\right),
\end{aligned}
$$

which is what you can also find in Problem 9.24 of Boas Chapter 7.



6. Boas Chapter 9, Problem 2.3

(Note: For Problem 2.3, you may want to utilize the indefinite integral found in Problem 13.19 of Boas Chapter 1 (— along with the formula in Problem 17.20 of Chapter 2), or in extensive references like Zwillinger (Sections 5.4.1 and 6.11.1, 33rd ed.).)

- The Euler equation, or its first integral ($\frac{\partial F}{\partial y'}$ = const.; Boas Chapter 9, Section 3) gives

$$
\begin{aligned}
\frac{\partial F}{\partial y} - \frac{\partial}{\partial x}\frac{\partial F}{\partial y'} = 0 = -\frac{\partial}{\partial x}\left(\frac{2xy'}{2\sqrt{1-y'^2}}\right) &\rightarrow\ x^2y'^2 = C_1^2(1-y'^2)\ \rightarrow\ y'^2 = \frac{C_1^2}{x^2+C_1^2} \\
&\rightarrow\ y = C_1\int\frac{d\left(\frac{dx}{C_1}\right)}{\sqrt{\left(\frac{x}{C_1}\right)^2+1}} \\
&\quad = C_1\ln\left[\frac{x}{C_1}+\sqrt{\left(\frac{x}{C_1}\right)^2+1}\right] + C_2 = \sinh^{-1}\frac{x}{C_1} + C_2.
\end{aligned}
$$

7. Boas Chapter 9, Problem 4.4

(Note: For Problem 4.4, as you evaluate the integral for T you may utilize the indefinite integral $\int \frac{dx}{\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{-a}}\sin^{-1}(\frac{2ax+b}{\sqrt{b^2-4ac}})$ (if $a<0$ and $b^2>4ac$) found in common integral tables like Appendix E of Thornton & Marion, or in extensive references like Zwillinger (Section 5.4.13, 33rd ed.).)

• With the gravitational potential energy inside the Earth in Problem 8.21 of Boas Chapter 6, the speed of the passenger train at r is found from

$$\frac{1}{2}mv^2 = \phi(R) - \phi(r) = -mgR - \frac{mg}{2R}(r^2 - 3R^2).$$

The integral representing the transit time,

$$\int dt = \int \frac{ds}{v} = \int \frac{\sqrt{1+r^2\theta'^2}}{\sqrt{\frac{g}{R}(R^2-r^2)}}dr,$$

is stationary when the Euler equation is satisfied, or a first integral of the Euler equation is constant. Thus,

$$\frac{\partial F}{\partial \theta'} = \frac{2r^2\theta'}{2\sqrt{\frac{g}{R}(R^2-r^2)}\sqrt{1+r^2\theta'^2}} = C \quad \to \quad \theta'^2 = \frac{\frac{g}{R}C^2(R^2-r^2)}{r^4 - \frac{g}{R}C^2r^2(R^2-r^2)},$$

where we can use $\frac{\partial r}{\partial \theta} = 0$ (or $\theta' = \frac{\partial \theta}{\partial r} = \infty$) at $r = r_0$ to evaluate the integration constant as

$$r_0^4 - \frac{g}{R}C^2r_0^2(R^2-r_0^2) = 0 \quad \to \quad C^2 = \frac{Rr_0^2}{g(R^2-r_0^2)}.$$

Plugging C back into the Euler equation, you find

$$1 + r^2\theta'^2 = 1 + \frac{\frac{g}{R}C^2(R^2-r^2)}{r^2 - \frac{g}{R}C^2(R^2-r^2)} = \frac{r^2(R^2-r_0^2)}{R^2(r^2-r_0^2)}$$

which leads to

$$\begin{aligned} T &= 2\int_{r_0}^{R} \frac{\sqrt{1+r^2\theta'^2}}{\sqrt{\frac{g}{R}(R^2-r^2)}}dr = 2\int_{r_0}^{R} \frac{\sqrt{r^2(R^2-r_0^2)}}{\sqrt{\frac{g}{R}(R^2-r^2)R^2(r^2-r_0^2)}}dr \\ &= \sqrt{\frac{R^2-r_0^2}{gR}}\int_{r_0}^{R} \frac{2rdr}{\sqrt{(R^2-r^2)(r^2-r_0^2)}} = \sqrt{\frac{R^2-r_0^2}{gR}}\left[-\sin^{-1}\left(\frac{-2r^2+(R^2+r_0^2)}{\sqrt{(R^2+r_0^2)^2-4R^2r_0^2}}\right)\right]_{r_0}^{R} \\ &= \sqrt{\frac{R^2-r_0^2}{gR}}\left[-\sin^{-1}\left(\frac{-R^2+r_0^2}{R^2-r_0^2}\right) + \sin^{-1}\left(\frac{R^2-r_0^2}{R^2-r_0^2}\right)\right] = \sqrt{\frac{R^2-r_0^2}{gR}}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \pi\sqrt{\frac{R^2-r_0^2}{gR}}. \end{aligned}$$

8. Boas Chapter 9, Problem 8.5

(Note: For Problem 8.5, you are asked to change the independent variable, as in Example 3 of Boas Chapter 9, Section 3.)

• As in Example 3 of Boas Chapter 9, Section 3, with $x' = \frac{dx}{dy} = \frac{1}{y'}$,

$$I = \int_{x_1}^{x_2} \sqrt{\frac{y'^2}{y^2} + 1}\, dx = \int_{y_1}^{y_2} \sqrt{\frac{y'^2}{y^2} + 1} \cdot x' dy = \int_{y_1}^{y_2} \sqrt{\frac{1}{y^2} + x'^2}\, dy.$$

Now, the Euler equation becomes

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x'} = 0 = \frac{\partial}{\partial y}\left(\frac{2x'}{2\sqrt{\frac{1}{y^2} + x'^2}}\right) \quad \to \quad \frac{1}{y^2} + x'^2 = C_1^2 x'^2 \quad \to \quad x' = \frac{1}{y\sqrt{C_1^2 - 1}}.$$

By integrating both sides, you get $x = \frac{1}{\sqrt{C_1^2 - 1}} \ln y + C_2$, or $y = Ae^{Bx}$.

9. In the class we discussed the resemblance between the heat flow equation and the Schrödinger equation, and left the latter for your exercise.

(a) Review how the *time-independent* Schrödinger equation, Eq.(3.22), and the time equation, Eq.(3.21), are acquired from the Schrödinger equation, Eq.(1.6), in Boas Chapter 13.

(b) Then, review the 1-dimensional "particle in a box" problem in Example 3 of Boas Chapter 13, Section 3. That is, starting from Eq.(3.23) with $V = 0$ on $(0, l)$ and boundary conditions $\Psi(0, t) = 0$ and $\Psi(l, t) = 0$, show how you can reach $\Psi(x, t)$ in Eq.(3.26) with $E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{l}\right)^2$.

(c) Now we consider an initial condition. Verify explicitly that an initial condition $\Psi(x, 0) = \frac{100}{l}x$ yields $\Psi(x, t)$ in Eq.(3.27). Repeat the exercise and find $\Psi(x, t)$ with a different initial condition $\Psi(x, 0) = 1$ on $(0, l)$.

(d) Here, we extend the problem in (b) to three dimensions. Show how your answer in (b) changes if you tackle the 3-dimensional "particle in a box" problem (e.g., a box with planar faces at $x = 0$, $x = l_x$, $y = 0$, $y = l_y$, $z = 0$, $z = l_z$). You are asked to explicitly separate the variables in Eq.(3.22) with $V = 0$, by assuming a solution of the form $\psi(x, y, z) = X(x)Y(y)Z(z)$, to reach

$$\Psi(x, y, z, t) = \sum_{n_x, n_y} A_{n_x n_y} \sin\frac{n_x \pi x}{l_x} \sin\frac{n_y \pi y}{l_y} \sin\frac{n_z \pi z}{l_z}\, e^{-iEt/\hbar},$$

where n_z is determined by $E = \frac{\hbar^2\pi^2}{2m}\left[\left(\frac{n_x}{l_x}\right)^2 + \left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2\right]$ for a given n_x, n_y, and E.

(e) Finally, we extend the problem in (b) to the case with a harmonic potential — i.e., a particle in a potential field $V = \frac{1}{2}kx^2$. Starting from the equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi,$$

(i.e., Eq.(3.23) of Boas Chapter 13 with $V = \frac{1}{2}kx^2$, or Eq.(2.45) of Griffiths & Schroeter) and changing the variable with $\xi = \alpha x = \left(\frac{mk}{\hbar^2}\right)^{\frac{1}{4}} \cdot x$ and $\lambda = \frac{2E}{\hbar}\left(\frac{m}{k}\right)^{\frac{1}{2}}$, demonstrate that $\psi(\xi)$ satisfies the equation,

$$-\frac{d^2\psi}{d\xi^2} + \xi^2\psi = \lambda\psi.$$

Then, assuming a solution of the form $\psi(\xi) = y(\xi)e^{-\xi^2/2}$, verify that $y(\xi)$ satisfies

$$y'' - 2\xi y' + (\lambda - 1)y = 0,$$

which is the Hermite equation seen in Eq.(22.14) of Boas Chapter 12, Section 22 (with $\lambda = 2n+1$), or in Eq.(2.79) of Griffiths & Schroeter. From this, prove that the possible wave functions for the harmonic oscillator potential are written as

$$\psi_n(\xi) = A_n H_n(\xi) e^{-\xi^2/2},$$

where $H_n(\xi)$ is the nth Hermite polynomial with the corresponding energy $E_n = \left(n+\frac{1}{2}\right)\hbar\left(\frac{k}{m}\right)^{\frac{1}{2}}$.

(Note: The Schrödinger equation discussed here should look familiar to most of you as you have studied the basic "particle in a box" problem in your elementary physics classes. If not, you may want to briefly review the freshman physics textbooks such as Halliday & Resnick, or the quantum mechanics textbooks such as Griffiths & Schroeter. For (e), briefly discuss how one could come up with the educated guess $\psi(\xi) = y(\xi)e^{-\xi^2/2}$. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

- (c) With a different initial condition $\Psi(x,0) = 1$ on $(0,l)$, you get

$$\Psi(x,t) = \sum_{\text{odd } n} \left(\frac{4}{n\pi}\right) \sin\frac{n\pi x}{l}\, e^{-iE_n t/\hbar}.$$

- (e-1) Since V approaches infinity as $x \to \infty$, the wave function there must approach zero. The wave function which behaves this way and satisfies the differential equation $-\frac{d^2\psi}{d\xi^2} + \xi^2\psi = 0$ (for $x^2 \gg \lambda$) should be in the form of $\psi(\xi) = y(\xi)e^{-\xi^2/2}$.

- (e-2) $\lambda = \lambda_n = 2n+1 = \frac{2E_n}{\hbar}\left(\frac{m}{k}\right)^{\frac{1}{2}}$ gives $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$ with $\omega = \left(\frac{k}{m}\right)^{\frac{1}{2}}$, a familiar result that can also be acquired with the ladder operator method (see Section 2.3.1 of Griffiths & Schroeter (3rd ed.)).

10. In the class we briefly discussed that the crucial properties of the Dirac delta function $\delta(x)$ may be developed as the limiting case using any of the following sequences of functions:

$$\delta_{n,1}(x) = \begin{cases} n, & \text{if } |x| < \frac{1}{2n} \\ 0, & \text{otherwise,} \end{cases} \qquad \delta_{n,2}(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2},$$

$$\delta_{n,3}(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}, \qquad \delta_{n,4}(x) = \frac{n}{2\cosh^2 nx},$$

$$\delta_{n,5}(x) = \frac{\sin nx}{\pi x}, \qquad \delta_{n,6}(x) = \frac{1}{2\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}, \quad \text{etc.}$$

In other words, we may regard $\delta(x)$ as a normalized *distribution* which is defined with

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx. \tag{1}$$

(a) Using $\delta_{n,1}(x)$ to $\delta_{n,5}(x)$, show that

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1 \quad \text{for all } n.$$

(b) Using $\delta_{n,1}(x)$ and $\delta_{n,5}(x)$, prove

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)f(x)dx = f(0).$$

(c) Treating $\delta_n(x)$ and its derivative as in Eq.(1), prove one of the properties of $\delta(x)$,

$$x\delta'(x) = -\delta(x).$$

(Note: For $\delta_{n,3}(x)$ in (a), you may need to prove and use the indefinite integral $\int \frac{dx}{1+x^2} = \arctan x$. For $\delta_{n,4}(x)$, you may simply utilize the indefinite integral $\int \operatorname{sech}^2 x\, dx = \tanh x$ found in common integral tables, or in extensive references like Zwillinger (Section 5.4.20, 33rd ed.). For $\delta_{n,5}(x)$, you may also take advantage of the definite integral $\int_{-\infty}^{\infty}\frac{\sin x}{x}dx = \pi$ found in Example 2 of Boas Chapter 7, Section 12 or in Example 4 of Chapter 14, Section 7.)

- (b) With an argument similar to the one right before Eq.(11.6) of Boas Chapter 8,

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_{n,1}(x)f(x)dx = \lim_{n\to\infty} n\int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x)dx = \lim_{n\to\infty} n\cdot\frac{f(\xi_n)}{n} = f(0)$$

where $\lim\limits_{n\to\infty}\xi_n = 0$. And

$$\begin{aligned}\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_{n,5}(x)f(x)dx &= \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin nx}{\pi x}f(x)dx = \lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{\sin y}{\pi(\frac{y}{n})}f\left(\frac{y}{n}\right)d\left(\frac{y}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin y}{y}f\left(\frac{y}{n}\right)dy = f(0).\end{aligned}$$

- (c) In a method similar to the one that checks Eq.(11.18b) in Boas Chapter 8,

$$\begin{aligned}\int_{-\infty}^{\infty}x\delta'(x)f(x)dx &= \lim_{n\to\infty}\int_{-\infty}^{\infty}x\delta_n'(x)f(x)dx = -\lim_{n\to\infty}\int_{-\infty}^{\infty}[xf(x)]'\cdot\delta_n(x)dx\\ &= -[xf(x)]'_{x=0} = -f(0) = -\lim_{n\to\infty}\int_{-\infty}^{\infty}\delta_n(x)f(x)dx = -\int_{-\infty}^{\infty}\delta(x)f(x)dx.\end{aligned}$$