Mathematical Physics I (Fall 2022): Homework #5 Solution

Due Nov. 18, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

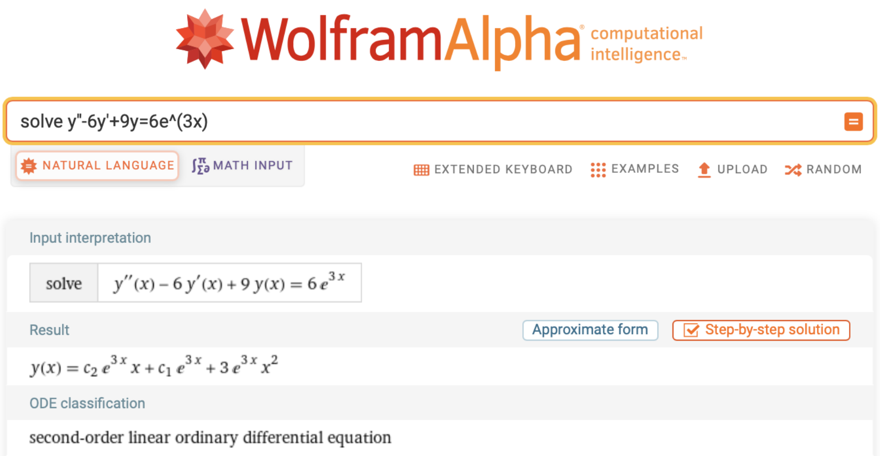
1. Boas Chapter 8, Problem 5.24

(Note: For Problems in Sections 5 and 6, read the instruction in the textbook carefully; that is, you have been asked to find a computer solution and reconcile differences, if any. For Problem 5.24, you will first want to review Problem 5.21.)

• The auxiliary equation of $(D+1)(D^2-D+1)y=0$ has three roots, -1 and $\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$. From Eqs.(5.16)-(5.18), you get $y=Ae^{-x}+e^{x/2}(Be^{i(\sqrt{3}/2)x}+Ce^{-i(\sqrt{3}/2)x})$, or $Ae^{-x}+e^{x/2}(D\sin\frac{\sqrt{3}}{2}x+E\cos\frac{\sqrt{3}}{2}x)$, or $Ae^{-x}+Fe^{x/2}\sin(\frac{\sqrt{3}}{2}x+\gamma)$.

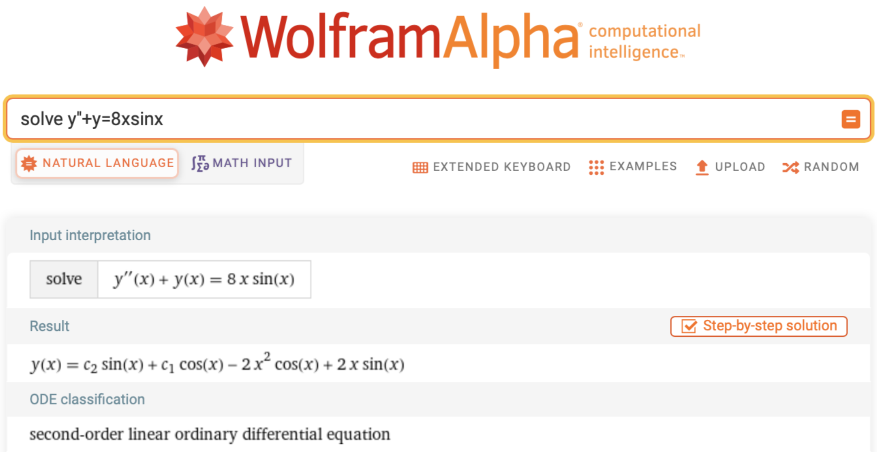
2. Boas Chapter 8, Problem 6.10

• From Eqs.(5.15) and (6.18), and with 3 as the equal root of the auxiliary equation of its homogeneous counterpart, you get $y=y_c+y_p=(Ax+B)e^{3x}+Cx^2e^{3x}$. Plugging y_p back into the equation yields $C=3$.



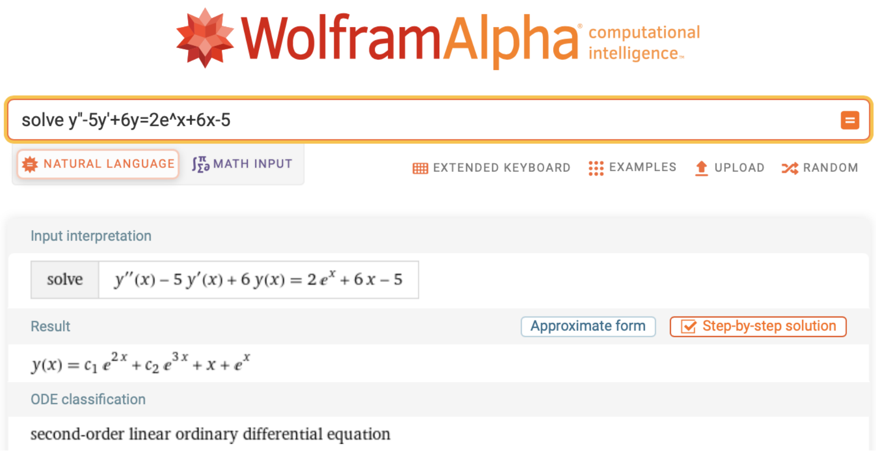
3. Boas Chapter 8, Problem 6.26

• Combining the technique in Examples 6 and 7 of Section 6, we attempt to find a particular solution of the equation $(D+i)(D-i)Y = 8xe^{ix}$. Because i equals to one of the roots of the auxiliary equation, from Eq.(6.24) you try a particular solution of the form $Y_p = xe^{ix}(Ax+B)$. Plugging Y_p back into the equation yields $A = -2i$, $B = 2$ and $Y_p = xe^{ix}(-2ix+2)$. Therefore, the particular solution we need is $y_p = Im(Y_p) = -2x^2 \cos x + 2x \sin x$.



4. Boas Chapter 8, Problem 6.34

• As in Example 9 of Section 6, $(D-2)(D-3)y = 2e^x + (6x-5)$ gives $y = y_c + y_{p1} + y_{p2} = Ae^{2x} + Be^{3x} + e^x + x$.



5. Boas Chapter 8, Problem 7.3

(Note: You may continue to utilize computer solutions to validate your answers to problems in Sections 7 to 9.)

• From Eq.(7.3) for Case (b), inserting $y' = p$ and $y'' = p\frac{dp}{dy}$ into the given differential equation gives $2y \cdot p\frac{dp}{dy} = p^2 \;\rightarrow\; \frac{2dp}{p} = \frac{dy}{y} \;\rightarrow\; \ln p^2 = \ln y + C_1 \;\rightarrow\; p^2 = C_2|y| = \left(\frac{dy}{dx}\right)^2 \;\rightarrow\; \frac{dy}{|y|^{1/2}} = C_3 dx$

$\rightarrow\ 2|y|^{\frac{1}{2}} = C_3 x + C_4 \ \rightarrow\ y = A(x+B)^2$. This seemingly "general" solution does not include an obvious solution by inspection, $y =$ constant, as we discussed in Example 3 of Section 2.

6. Boas Chapter 8, Problem 7.22

• From Eqs.(7.17)-(7.19) for Case (d), inserting $xy' = \frac{dy}{dz}$ and $x^2y'' = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ into the given differential equation gives $\frac{d^2y}{dz^2} + y = 2x = 2e^z$, which now became identical to Problem 6.5. Thus, from $y(z) = y_c + y_p = Ae^{iz} + Be^{-iz} + e^z$ or $C\cos z + D\sin z + e^z$, you get $y(x) = C\cos(\ln x) + D\sin(\ln x) + x$.

7. Boas Chapter 8, Problem 8.12

• In order to utilize $L9$-$L10$ in the Laplace transform table (Boas p.469-471), we write

$$Y = \frac{3p+10}{p^2-25} = 3\cdot\frac{p}{p^2-25} + 2\cdot\frac{5}{p^2-25},$$

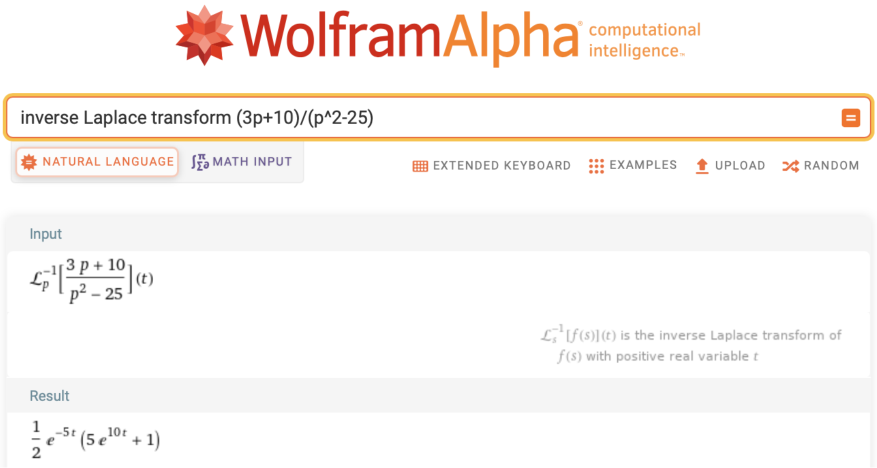
from which you can find $y = L^{-1}(Y) = 3\cosh 5t + 2\sinh 5t$, or $\frac{5}{2}e^{5t} + \frac{1}{2}e^{-5t}$.

• Alternatively, to utilize $L7$-$L8$ in the Laplace transform table, we reshape the given form as

$$Y = \frac{3p+10}{p^2-25} = \frac{3p+10}{(p-5)(p+5)} = 3 \cdot \frac{p}{(p-5)(p+5)} + 10 \cdot \frac{1}{(p-5)(p+5)}.$$

Therefore, $y = L^{-1}(Y)$ is found to be

$$y = 3 \cdot \frac{-5e^{5t} - 5e^{-5t}}{-5-5} + 10 \cdot \frac{e^{5t} - e^{-5t}}{5-(-5)} = \frac{3}{2}e^{5t} + \frac{3}{2}e^{-5t} + e^{5t} - e^{-5t} = 3\cosh 5t + 2\sinh 5t.$$

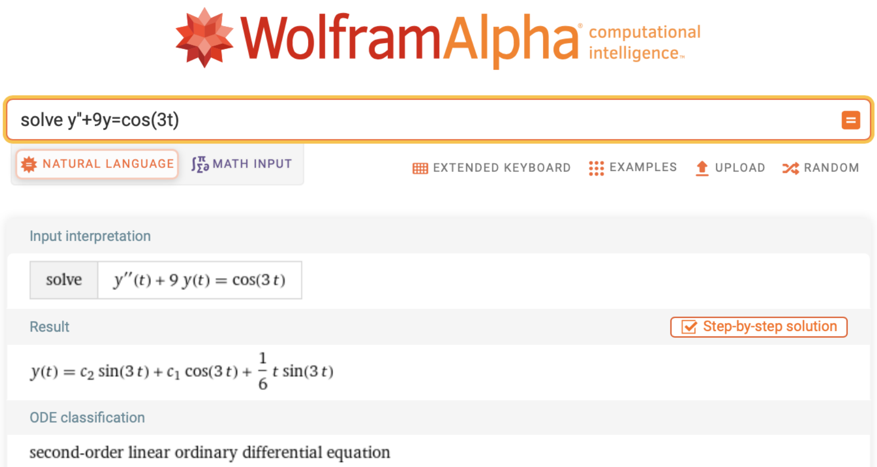


8. Boas Chapter 8, Problem 9.16

• With Eqs.(9.1)-(9.2) and $L4$ in the Laplace transform table, we transform the given differential equation into

$$p^2Y - py_0 - y_0' + 9Y = L(\cos 3t) \;\rightarrow\; p^2Y - 2p + 9Y = \frac{p}{p^2+3^2} \;\rightarrow\; Y = \frac{p}{(p^2+3^2)^2} + \frac{2p}{p^2+3^2},$$

from which you get $y = L^{-1}(Y) = \frac{1}{6}t\sin 3t + 2\cos 3t$ by using $L4$ and $L11$ in the Laplace transform table.



9. Let us apply the Laplace transform to two examples in nuclear physics.

(a) Consider a series of radioactive decays between three nuclides, with Nuclide 1 decaying into Nuclide 2, and Nuclide 2 into Nuclide 3. The concentration fo the nuclides satisfy the system of differential equations

$$\frac{dN_1}{dt} = -\lambda_1 N_1, \quad \frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2, \quad \frac{dN_3}{dt} = \lambda_2 N_2,$$

where one can see that Nuclide 3 is assumed to be stable. λ_1 and λ_2 are decay constants. Explain the meaning of each term in each equation. Then, with initial conditions $N_1(0) = N_0$, $N_2(0) = 0$, and $N_3(0) = 0$, find $N_1(t)$, $N_2(t)$, and $N_3(t)$. Tackle the problem in two different ways, first by using the methods described in Boas Chapter 8, Sections 2 and 3, then by using the Laplace transform.

(b) Now consider a different type of radioactive decay seen in a nuclear reactor. The rate of change in concentration of Nuclide 2 is now modified to

$$\frac{dN_2}{dt} = \phi(\sigma_1 N_0 - \sigma_2 N_2) - \lambda_2 N_2,$$

where one can see that the concentration of Nuclide 1 is assumed to be (approximately) constant — i.e., $N_1(t) = N_1(0) = N_0$. ϕ is the neutron flux (in $\text{cm}^{-2}\,\text{s}^{-1}$) and σ_1 and σ_2 (in cm^2) are neutron absorption cross sections. Briefly explain only in words the meaning of each term in the equation. Then, with an initial condition $N_2(0) = 0$, find $N_2(t)$.

(Note: If you are not familiar with the topics discussed here such as radioactive decay or a nuclear reactor, you may review the freshman physics textbooks such as Halliday & Resnick. For (a), you may also want to review Example 2 in Boas Chapter 8, Section 3. For (b), find the numerical value of N_2 at $t = 1$ year, the concentration of ^{154}Eu which the original isotope ^{153}Eu is decaying into, using the following constants: $\sigma_1 = 4 \times 10^{-22}\text{cm}^2$, $\sigma_2 = 10^{-21}\text{cm}^2$, $\lambda_2 = 1.4 \times 10^{-9}\text{s}^{-1}$, $\phi = 10^9\text{cm}^{-2}\,\text{s}^{-1}$, and $N_0 = 10^{20}$. Check if the assumption that $N_1(t) = N_0$ is justified. Control rods in nuclear reactors should be made of elements capable of absorbing many neutrons without themselves decaying, such as ^{153}Eu.)

• (a-1) As in Example 2 in Boas Chapter 8, Section 3, you can use Eq.(3.9) to acquire the desired equations.

• (a-2) Using Eq.(9.1) of Boas Chapter 8, you get $pY_1 + \lambda_1 Y_1 = N_0$, which has the solution $Y_1 = \frac{N_0}{p+\lambda_1}$. From $L2$ of the Laplace transform table (Boas p.469-471), you reach $N_1(t) = N_0 e^{-\lambda_1 t}$. Then, from the second equation you get $pY_2 + \lambda_2 Y_2 = \frac{N_0\lambda_1}{p+\lambda_1}$, which has the solution $Y_2 = \frac{N_0\lambda_1}{(p+\lambda_1)(p+\lambda_2)}$. $L7$ of the Laplace transform table gives us $N_2(t) = N_0\lambda_1 \frac{e^{-\lambda_1 t} - e^{-\lambda_2 t}}{\lambda_2 - \lambda_1}$.

• (b) The first two terms on the right-hand side describe the production and destruction of Nuclide 2 via neutron absorption. The last term describes the radioactive decay of Nuclide 2. Using Eq.(9.1) of Boas Chapter 8, you can easily get $N_2(t) = \frac{N_0\phi\sigma_1}{\lambda_2+\phi\sigma_2}\left(1 - e^{-(\lambda_2+\phi\sigma_2)t}\right)$.

10. In the beginning of Boas Chapter 8, Section 7, the author discusses many methods of solving various types of second-order ODEs. Among them is Lagrange's *method of variation of parameters* to find a particular solution of an inhomogeneous ODE.

(a) Let us start with a homogeneous second-order linear ODE in the form of

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous functions of x. Let us assume that we know its two independent solutions, y_1 and y_2. Now, for the inhomogeneous second-order linear ODE of

$$y'' + p(x)y' + q(x)y = f(x),$$

show that a particular solution $y_p(x)$ is written as

$$y_p(x) = -y_1(x)\int \frac{y_2(x')f(x')}{W(x')}dx' + y_2(x)\int \frac{y_1(x')f(x')}{W(x')}dx'$$

where $W(x')$ is the Wronskian of y_1 and y_2, $W(y_1(x'), y_2(x'))$.

(Note: You may start with $y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$ and follow the step-by-step instruction given in Boas Chapter 8, Problem 12.14(b) that leads to the set of two conditions for c_1 and c_2: $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ and $c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = f(x)$. Notice that the first equation of this set is our "imposed" condition, while the second one is what you get if you plug $y_p' = \{c_1'(x)y_1(x) + c_2'(x)y_2(x)\} + \{c_1(x)y_1'(x) + c_2(x)y_2'(x)\} = c_1(x)y_1'(x) + c_2(x)y_2'(x)$ and the corresponding y_p'' into our ODE above. In case you wonder, no knowledge about the Green function in Section 12 is needed to tackle this problem.)

Now, utilizing the given solution of the homogeneous equation, find a solution of each of the following inhomogeneous ODEs. (More exercise problems in Chapter 8, Problems 12.15-18.)

(b) $y'' + y = \sec x$; with $y_1 = \cos x$ and $y_2 = \sin x$

(c) $(1-x)y'' + xy' - y = (1-x)^2$; with $y_1 = x$ and $y_2 = e^x$

- (b) $y(x) = C_1 \cos x + C_2 \sin x + y_p(x) = (C_1 + \ln|\cos x|)\cos x + (C_2 + x)\sin x$
- (c) $y(x) = C_1 x + C_2 e^x + y_p(x) = C_1' x + C_2 e^x + x^2 + 1$