Mathematical Physics I (Fall 2022): Homework #4 Solution

Due Nov. 4, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

1. Boas Chapter 7, Problem 5.4

(Note: For Problem 5.4, you are asked to tackle the problem in two ways — first directly find the Fourier coefficients, then verify your result using the answer given in Problem 5.3.)

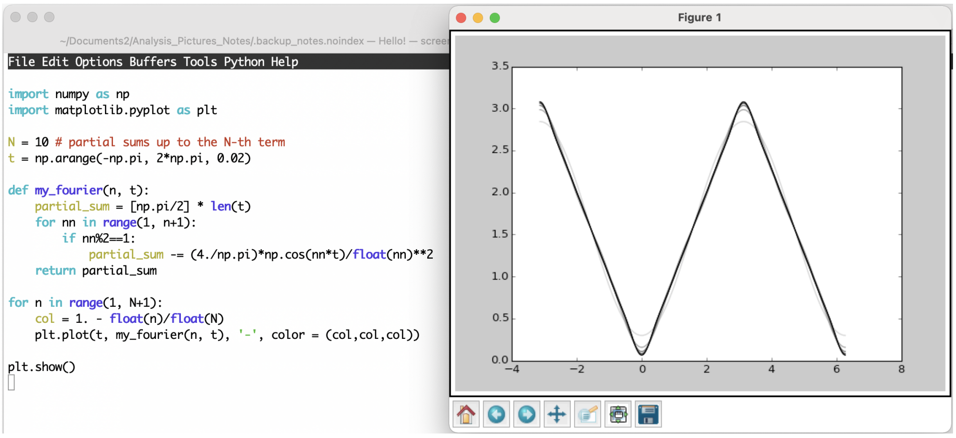
- From Eqs.(5.6), (5.9)-(5.10), you can find

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left\{-\int_{-\pi}^{\pi/2} dx + \int_{\pi/2}^{\pi} dx\right\} = -1,\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left\{-\int_{-\pi}^{\pi/2}\cos nx\, dx + \int_{\pi/2}^{\pi}\cos nx\, dx\right\}\\
&= \frac{1}{\pi}\left\{-\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi/2} + \left[\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}\right\} = -\frac{2\sin(n\pi/2)}{n\pi} = \begin{cases} 0, & \text{if } n \text{ even,}\\ \frac{2(-1)^n}{n\pi}, & \text{if } n \text{ odd,}\end{cases}\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left\{-\int_{-\pi}^{\pi/2}\sin nx\, dx + \int_{\pi/2}^{\pi}\sin nx\, dx\right\}\\
&= \frac{1}{\pi}\left\{-\left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi/2} + \left[-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}\right\} = \frac{2\cos(n\pi/2)}{n\pi} - \frac{2\cos n\pi}{n\pi} = \begin{cases} \frac{2}{n\pi}, & \text{if } n \text{ odd,}\\ 0, & \text{if } n = 4k,\\ -\frac{4}{n\pi}, & \text{if } n = 4k+2.\end{cases}
\end{aligned}
$$

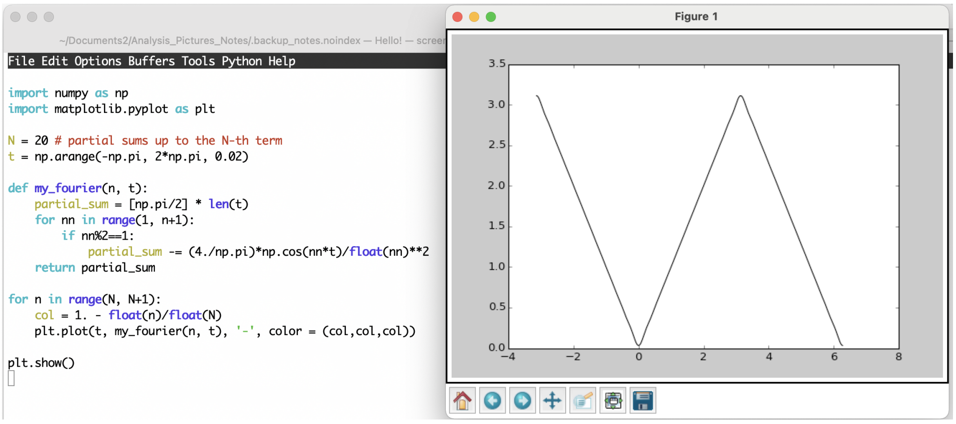
2. Boas Chapter 7, Problem 6.12 (for Problem 5.9)

(Note: For Problems 6.12 and 7.9, you will first have to work out Problem 5.9. You are asked to tackle Problem 5.9 in two ways — first directly find the Fourier coefficients, then verify your result using the answer given in Problem 5.7. Then for Problem 6.12 you may consider a way to automatically draw several partial sums of different n's — e.g., a simple script written in MATLAB or in `python`; however, making such an automated script is not necessary for you to receive a full credit.)

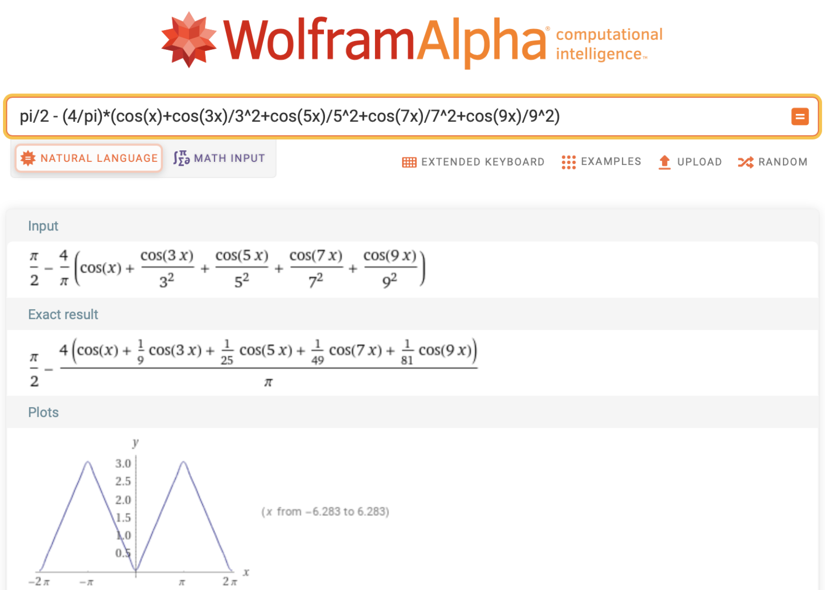
- An example of an automated plotting script ($n = 1$ to 9) is as follows:



By pushing n to 20, a very good approximation of the given $f(x)$ is acquired already:



• You are, however, not required to come up with an automated script like above. You may simply use e.g., WolframAlpha to create multiple plots in a row. For example, the black line $(n = 9)$ in the first plot above matches what is seen below:



3. Boas Chapter 7, Problem 7.9

• From Eqs.(7.4) and (7.6), you can find

$$
\begin{aligned}
c_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left\{-\int_{-\pi}^{0} xdx + \int_0^{\pi} xdx\right\} = \frac{\pi}{2},\\
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\left\{-\int_{-\pi}^{0} xe^{-inx}dx + \int_0^{\pi} xe^{-inx}dx\right\}\\
&= \frac{1}{2\pi}\left\{-\left[\frac{xe^{-inx}}{-in}\right]_{-\pi}^{0} - \frac{1}{in}\int_{-\pi}^{0} e^{-inx}dx + \left[\frac{xe^{-inx}}{-in}\right]_0^{\pi} + \frac{1}{in}\int_0^{\pi} e^{-inx}dx\right\}\\
&= \frac{1}{2\pi}\cdot 2\cdot\frac{(-1)^n-1}{n^2} = \begin{cases} 0, & \text{if } n \text{ even,}\\ -\frac{2}{\pi n^2}, & \text{if } n \text{ odd.}\end{cases}
\end{aligned}
$$

4. Boas Chapter 7, Problem 9.20

(Note: For Problem 9.20, check out Example in Boas Chapter 7, Section 9 for a worked example.)

• For even function f_c of period $2b$, from Eqs.(5.6), (5.9)-(5.10) or (8.3) or (9.5), you can find $b_n = 0$,

$$
a_0 = \frac{1}{1}\int_{-1}^{1} x^2dx = 2\int_0^1 x^2dx = \frac{2}{3},
$$

and

$$
\begin{aligned}
a_n &= 2\int_0^1 x^2\cos n\pi x\,dx = 2\left\{\left[\frac{x^2\sin n\pi x}{n\pi}\right]_0^1 - \frac{2}{n\pi}\int_0^1 x\sin n\pi x\,dx\right\}\\
&= -\frac{4}{n\pi}\left\{-\left[\frac{x\cos n\pi x}{n\pi}\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos n\pi x\,dx\right\} = \frac{4\cos n\pi}{n^2\pi^2} = \frac{4(-1)^n}{n^2\pi^2}.
\end{aligned}
$$

5. Boas Chapter 7, Problem 12.26

(Note: For Problem 12.26, you are asked to first work out Problem 12.15.)

• From Eq.(12.15), you can find

$$
\begin{aligned}
g_c(\alpha) &= \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\alpha x\,dx = \sqrt{\frac{2}{\pi}}\int_0^{a}(-2x+2a)\cos\alpha x\,dx\\
&= -\sqrt{\frac{2}{\pi}}\int_0^a 2x\cos\alpha x\,dx + \sqrt{\frac{2}{\pi}}\left[\frac{2a\sin\alpha x}{\alpha}\right]_0^a\\
&= -\sqrt{\frac{2}{\pi}}\left[\frac{2x\sin\alpha x}{\alpha}\right]_0^a + \sqrt{\frac{2}{\pi}}\int_0^a \frac{2\sin\alpha x}{\alpha}dx + \sqrt{\frac{2}{\pi}}\frac{2a\sin\alpha a}{\alpha} = 2\sqrt{\frac{2}{\pi}}\cdot\frac{1-\cos\alpha a}{\alpha^2},
\end{aligned}
$$

which gives

$$
f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} g_c(\alpha)\cos\alpha x\,d\alpha = \frac{4}{\pi}\int_0^{\infty}\frac{1-\cos\alpha a}{\alpha^2}\cos\alpha x\,d\alpha.
$$

From the Fourier integral theorem in Boas Chapter 7, Section 12, and with $a = 1$, you reach

$$2 = f(0) = \frac{4}{\pi}\int_0^\infty \frac{1-\cos\alpha}{\alpha^2}\,d\alpha.$$

6. Boas Chapter 7, Problem 13.4

(Note: For Problem 13.4(a), use the technique discussed in Boas Chapter 8, Section 2.)

- (c) Writing $q(t) = q_p + q_c(t) = CV - CVe^{-t/RC}$, we first aim to expand $q_c(t) = -CVe^{-t/RC}$ in a complex exponential Fourier series. From Eqs.(8.2)-(8.3) with $2l = \frac{1}{2}RC$, you can write

$$\begin{aligned}
q_c(t) &= \sum_{n=-\infty}^{\infty} c_n e^{4in\pi t/RC}, \quad \text{and} \\
c_0 &= \frac{2}{RC}\int_0^{RC/2} q_c(t)dt = -\frac{2CV}{RC}\int_0^{RC/2} e^{-t/RC}dt = -2CV(1-e^{-1/2}), \\
c_n &= \frac{2}{RC}\int_0^{RC/2} q_c(t)e^{-4in\pi t/RC}dt = -\frac{2CV}{RC}\int_0^{RC/2} e^{-t/RC}e^{-4in\pi t/RC}dt \\
&= -2CV\left(\frac{1-e^{-(\frac{1}{2}+2in\pi)}}{1+4in\pi t}\right) = -2CV\left(\frac{1-e^{-1/2}}{1+4in\pi t}\right).
\end{aligned}$$

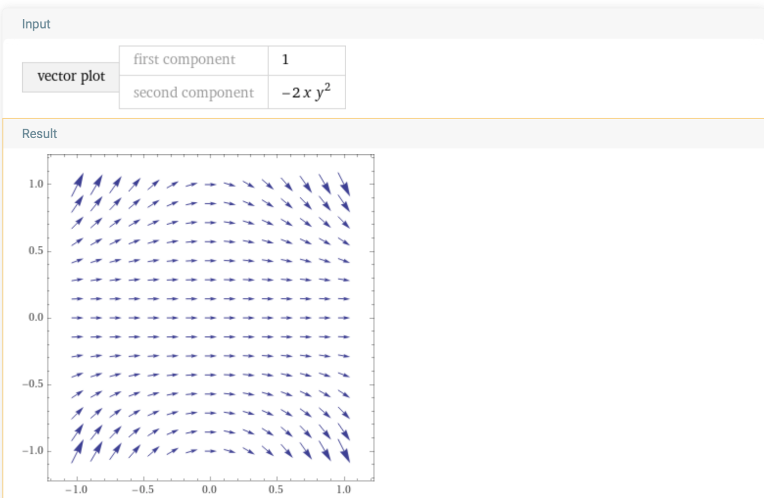
Thus, you can express $q(t)$ as

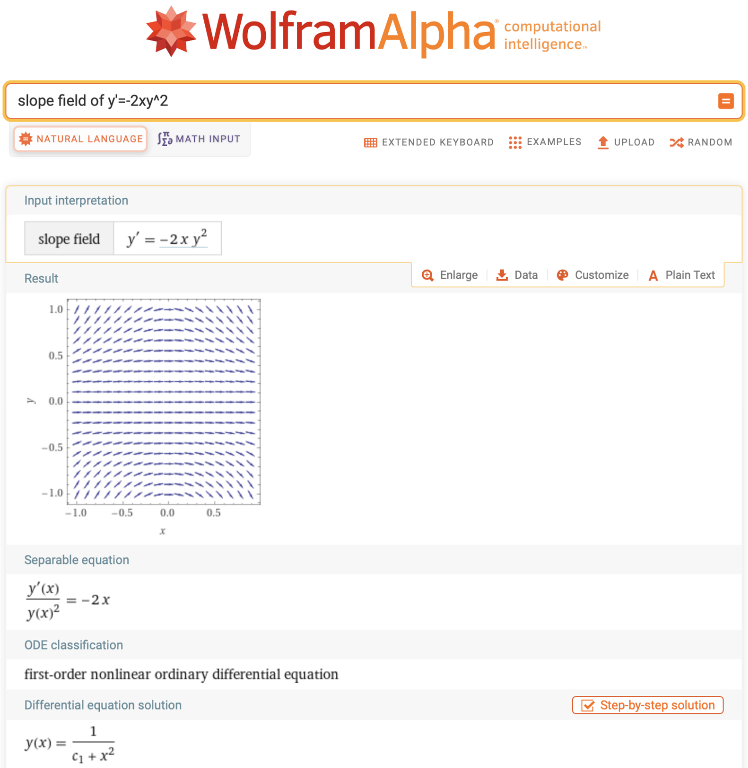
$$q(t) = q_p + q_c(t) = CV\left\{1 - 2(1-e^{-1/2})\sum_{n=-\infty}^{\infty}\frac{e^{4in\pi t/RC}}{1+4in\pi t}\right\}.$$

7. Boas Chapter 8, Problem 2.8

(Note: For Problem 2.8, read the instruction in the textbook carefully; that is, you have been asked to plot a slope field with a computer, for example. Read Boas Chapter 8, Section 1 to learn about the "slope field".)

- $\frac{dy}{y^2} = -2xdx \;\rightarrow\; -\frac{1}{y} = -x^2 + C_1 \;\rightarrow\; y = \frac{1}{x^2+C} \;\rightarrow\; (x, y) = (2, 1)$ gives $C = -3$.



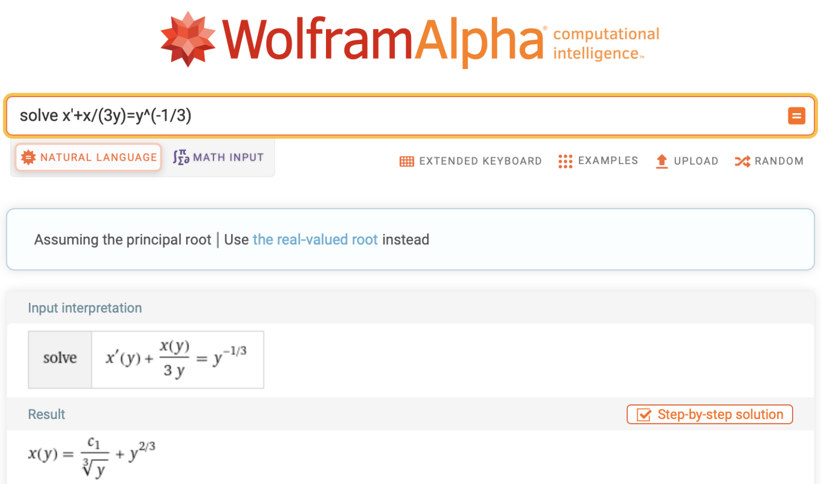


8. Boas Chapter 8, Problem 3.14

(Note: For Problem 3.14, again, read the instruction and the hint in the textbook carefully.)

• $x' + \frac{1}{3y}x = y^{-\frac{1}{3}} \ \rightarrow\ \frac{dx}{x} = -\frac{dy}{3y} \ \rightarrow\ \ln x = \ln y^{-\frac{1}{3}} + C_1 \ \rightarrow\ x_c = Cy^{-\frac{1}{3}}$. Meanwhile, plugging $x_p = Ay^{\frac{2}{3}}$ in the original equation, you get $A = 1$. Therefore, $x = x_c + x_p = Cy^{-\frac{1}{3}} + y^{\frac{2}{3}}$.

• Alternatively, from Eqs.(3.4) and (3.9), $I = \int P dy = \int \frac{dy}{3y} = \frac{1}{3}\ln y \ \rightarrow\ x = e^{-I}\int Qe^{I} dy + Ce^{-I} = y^{-\frac{1}{3}}\int y^{-\frac{1}{3}} y^{\frac{1}{3}} dy + Cy^{-\frac{1}{3}} = y^{\frac{2}{3}} + Cy^{-\frac{1}{3}}$.



9. Let us use the Fourier transform to appreciate the meaning of the Heisenberg uncertainty principle in quantum mechanics.

(a) Imagine an infinite wave train $\sin\omega_0 t$ clipped by shutters to maintain only N cycles of the original waveform:

$$f(t) = \begin{cases} \sin\omega_0 t, & \text{if } |\omega_0 t| < N\pi, \\ 0, & \text{if } |\omega_0 t| > N\pi. \end{cases}$$

Find the amplitude function of the Fourier (exponential) transform, $g(\omega)$. Since the prefactor may depend on the exact definition of the transform, do not worry too much about it.

(b) Find the amplitude function of the Fourier sine transform, $g_s(\omega)$. Again, since the prefactor may depend on the exact definition of the transform, do not worry too much about it.

(c) Show that, in the special case of $N = \frac{1}{2}$ and $\omega_0 = 1$, $g_s(\omega)$ becomes

$$g_{s,1}(\omega) = \sqrt{\frac{2}{\pi}} \frac{\omega\cos\left(\frac{\omega\pi}{2}\right)}{1-\omega^2}.$$

(d) Now consider the limit of $\omega_0 \gg 1$ and $\omega \approx \omega_0$. Show that $g_s(\omega)$ is approximately equal to

$$g_{s,2}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{\sin[(\omega_0-\omega)\frac{N\pi}{\omega_0}]}{\omega_0-\omega}.$$

Sketch or computer plot this function for $N = 1, 3, 5, 10$ and an arbitrary ω_0. Notice that for $N \gg 1$, $g_{s,2}(\omega)$ may be interpreted as proportional to what we later define as the Dirac delta function, δ, in Boas Chapter 8, Section 11.

(Note: You do not need to provide a mathematically rigorous proof that $g_{s,2}(\omega)$ indeed becomes proportional to $\delta(\omega-\omega_0)$ as $N \to \infty$. For now, observe the shape of $g_{s,2}(\omega)$ as you vary N, and based on your observation simply argue that $\lim_{\omega\to\omega_0} g_{s,2}(\omega) \to \infty$ as $N \to \infty$.)

(e) Show that the first zeros of $g_{s,2}(\omega)$ from $\omega = \omega_0$ are at $\omega = \omega_0 \pm \Delta\omega = \omega_0 \pm \frac{\omega_0}{N}$. Justify that $\Delta\omega = \frac{\omega_0}{N}$ could be a good measure of the spread (or uncertainty) in frequency of our clipped wave train. Then, establish the inverse relationship between the wave train's pulse length ($N\pi$) and the frequency spread ($\Delta\omega$). Finally, using the relationship along with the assumed *wave nature of matter*, explain the uncertainty principle of quantum mechanics.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. Note that the inverse relationship found here is a fundamental property of the finite wave train, and has little to do with any additional *ad hoc* postulates in quantum mechanics. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

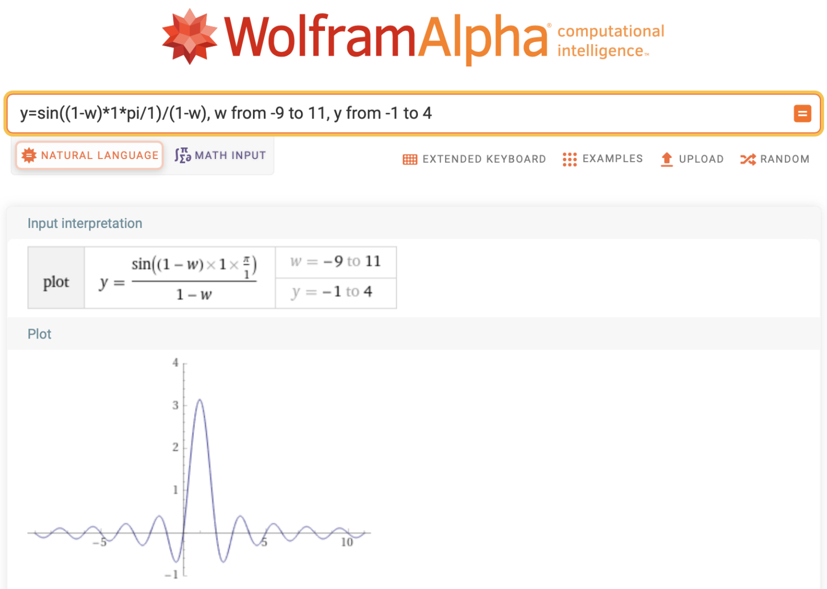
• (a) From Eq.(12.2) of Boas Chapter 7, you find

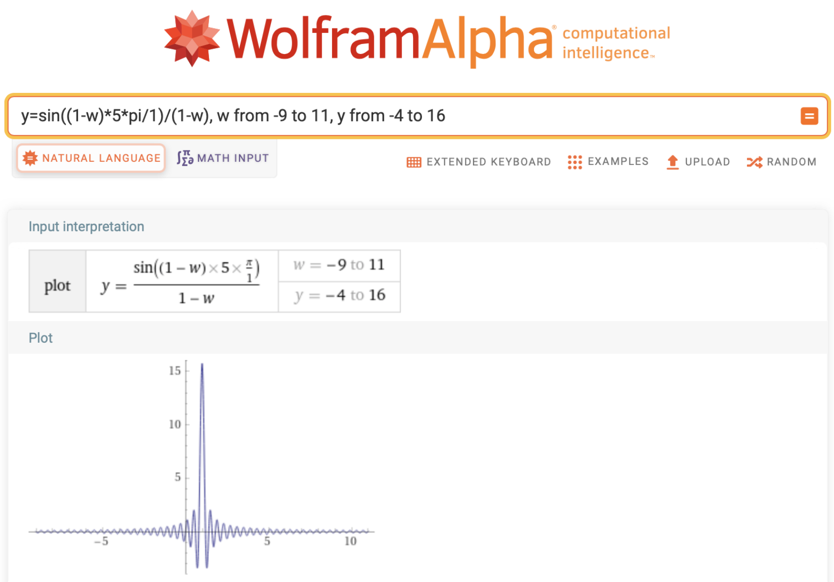
$$\begin{aligned} g(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-\frac{N\pi}{\omega_0}}^{\frac{N\pi}{\omega_0}} \sin\omega_0 t\; e^{-i\omega t}dt = \frac{1}{2\pi}\int_{-\frac{N\pi}{\omega_0}}^{\frac{N\pi}{\omega_0}} \frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i}\cdot e^{-i\omega t}dt \\ &= \frac{1}{2\pi i}\left[\frac{e^{i(\omega_0-\omega)\frac{N\pi}{\omega_0}} - e^{-i(\omega_0-\omega)\frac{N\pi}{\omega_0}}}{2i(\omega_0-\omega)} - \frac{e^{i(\omega_0+\omega)\frac{N\pi}{\omega_0}} - e^{-i(\omega_0+\omega)\frac{N\pi}{\omega_0}}}{2i(\omega_0+\omega)}\right] \\ &= \frac{1}{2\pi i}\left[\frac{\sin[(\omega_0-\omega)\frac{N\pi}{\omega_0}]}{\omega_0-\omega} - \frac{\sin[(\omega_0+\omega)\frac{N\pi}{\omega_0}]}{\omega_0+\omega}\right]. \end{aligned}$$

• (b) From Eq.(12.14) of Boas Chapter 7, you can find

$$g_s(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\frac{N\pi}{\omega_0}} \sin \omega_0 t \, \sin \omega t \, dt = \frac{1}{\sqrt{2\pi}} \left[\frac{\sin[(\omega_0 - \omega)\frac{N\pi}{\omega_0}]}{\omega_0 - \omega} - \frac{\sin[(\omega_0 + \omega)\frac{N\pi}{\omega_0}]}{\omega_0 + \omega} \right].$$

• (d) $\lim_{\omega \to \omega_0} g_{s,2}(\omega) = \frac{1}{\sqrt{2\pi}} \frac{N\pi}{\omega_0} \to \infty$ as $N \to \infty$. Computer plots of $g_{s,2}(\omega)$ for $N = 1$ and 5 can be found below.





10. Let us consider various objects falling under the gravitational acceleration g.

(a) Consider a ball of mass m falling downward. y is the distance the ball traveled at time t (i.e., $y > 0$ and $\hat{\mathbf{e}}_y$ points downward). The ball experiences a resistive force proportional to its speed, $-bv(t)$, where $v(t) = y'(t)$. With the initial condition $v(0) = 0$, show that the speed of the ball is written as

$$v(t) = v_{\rm t}\left(1 - e^{-\frac{b}{m}t}\right),$$

where $v_{\rm t} = mg/b$ is a terminal velocity as $t \to \infty$.

(b) Now consider a different air drag. A falling parachutist experiences a quadratic resistive force $-bv^2(t)$ on the parachute. For simplicity, assume that the parachute opens immediately at $t = 0$ when $v(0) = 0$. Prove that the speed of the parachutist can be written as

$$v(t) = v_{\rm t}\tanh\left(\frac{t}{T}\right),$$

where $v_{\rm t} = \sqrt{mg/b}$ is now a different terminal velocity and $T = \sqrt{m/gb}$ is the timescale that characterizes the asymptotic approach of $v(t)$ to $v_{\rm t}$.

(c) Let us insert numerical values in your answer in (b). For a skydiver in free fall with $m = 70$ kg, find the terminal velocity with the constant of proportionality (friction coefficient) $b = 0.25\,\mathrm{kg\,m^{-1}}$. Then, for the same skydiver but now with her parachute open, find the terminal velocity with $b = 700\,\mathrm{kg\,m^{-1}}$.

(Note: If you are not familiar with the concept of terminal velocity, you may want to review the freshman physics textbooks such as Halliday & Resnick. For (b), you may find it useful to write your equation of motion in a separable form, $\frac{dv}{v_{\rm t}^2 - v^2} = \frac{bdt}{m}$.)

• (a-1) The equation of motion $F = mv' = mg - bv$ (with both $\hat{\mathbf{e}}_y$ and $\mathbf{g}$ pointing downward) $\to\ \frac{dv}{g - \frac{b}{m}v} = dt\ \to\ -\frac{m}{b}\ln\left(g - \frac{b}{m}v\right) = t + C_1\ \to\ v(t) = \frac{mg}{b} - \frac{mC_2}{b}e^{-\frac{b}{m}t} = v_{\rm t}\left(1 - e^{-\frac{b}{m}t}\right)$.

• (a-2) Alternatively, you may solve this problem using the Laplace transform discussed in Boas Chapter 8, Section 8. Taking the Laplace transform of the equation with $Y = L(v)$ and $v(0) = 0$, you find $mpY = \frac{mg}{p} - bY\ \to\ Y = \frac{g}{p(p+\frac{b}{m})} = \frac{mg}{b}\left(\frac{1}{p} - \frac{1}{p+\frac{b}{m}}\right)\ \to\ v = v_{\rm t}\left(1 - e^{-\frac{b}{m}t}\right)$.

• (b) $\frac{dv}{v_{\rm t}^2 - v^2} = \frac{bdt}{m}\ \to\ \frac{1}{2v_{\rm t}}\ln\left(\frac{v_{\rm t}+v}{v_{\rm t}-v}\right) = \frac{bt}{m}\ \to\ v(t) = v_{\rm t}\frac{\exp\left(\frac{2t}{T}\right)-1}{\exp\left(\frac{2t}{T}\right)+1} = v_{\rm t}\tanh\left(\frac{t}{T}\right)$.