Mathematical Physics I (Fall 2022): Homework #3 Solution

Due Oct. 14, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

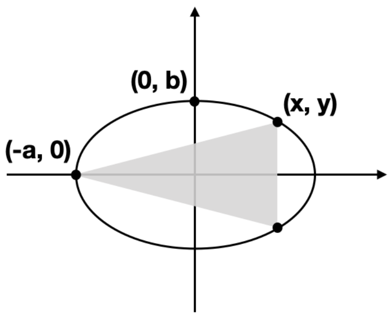
1. Boas Chapter 4, Problem 7.28

(Note: For Problem 7.28, you may want to prove and utilize the findings in Problem 7.27. Note the meaning of the subscripts next to the partial derivatives from Boas Chapter 4, Section 1 in case you did not read it. Prove further that the resulting formula is the familiar $c_p - c_v = nR$ found in e.g., Schroeder or Halliday & Resnick, where n is the number of moles of gas present and R is the gas constant.)

• Replacing $x \to T$, $y \to v$, $z \to p$ and $u \to s$ gives $\left(\frac{\partial s}{\partial T}\right)_p = \left(\frac{\partial s}{\partial T}\right)_v + \left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_p$, thus $c_p = c_v + T\left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_p$. With $\left(\frac{T\partial s}{\partial v}\right)_T = p$ and $\left(\frac{\partial v}{\partial T}\right)_p = \left(\frac{\partial (nRT/p)}{\partial T}\right)_p = \frac{nR}{p}$, you find $c_p - c_v = nR$.

2. Boas Chapter 4, Problem 9.9

(Note: For Problem 9.9, tackle the problem in two ways, first with the substitution method in Section 8 and then with the Lagrange multiplier method in Section 9.)



• With $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we can write the square of the triangle's area A as $A^2 = (x+a)^2 y^2 = (x+a)^2 b^2 (1 - \frac{x^2}{a^2})$. Differentiating A^2 with x, $\frac{\partial A^2}{\partial x} = 0$ gives $2b^2(2x-a)(x+a) = 0$.

• Alternatively, to utilize the Lagrange multiplier, we write $F = A + \lambda\phi$ from Eq.(9.6) with $A = (x+a)y$ and $\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2}$. Differentiating F with two variables x and y, $\frac{\partial F}{\partial x} = 0$ gives $y + \frac{2\lambda}{a^2} \cdot x = 0$, $\frac{\partial F}{\partial y} = 0$ gives $x + \frac{2\lambda}{b^2} \cdot y = -a$. Solving the set of two equations using the Cramer's rule you get $x = \frac{a}{\frac{4\lambda^2}{a^2b^2}-1}$ and $y = \frac{(-2\lambda/a)}{\frac{4\lambda^2}{a^2b^2}-1}$, which can be plugged into the constraint $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to get $\lambda = \pm\frac{3\sqrt{3}ab}{2}$, $x = \frac{a}{2}$, and $y = \mp\frac{\sqrt{3}b}{2}$. Hence, $A = (x+a)y = \frac{3\sqrt{3}ab}{4}$.

3. Boas Chapter 4, Problem 11.7

(Note: For Problem 11.7, notice that the equation we start with is a special case of Eq.(2.1) in Boas Chapter 12, Section 2.)

• By inserting $\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = -\frac{1}{\sin\theta}\frac{dy}{d\theta}$ and $\frac{d^2y}{dx^2} = -\frac{\cos\theta}{\sin^3\theta}\frac{dy}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2y}{d\theta^2}$ into the given Legendre equation, one can easily acquire the desired expression written in y and θ.

4. Boas Chapter 5, Problem 4.19

(Note: For Problem 4.19, you may want to prove and utilize the first theorem in Problem 4.18. You will also have to show that the intervals of integration for u and v are $[-\infty, \infty]$ and $[0, \infty]$, respectively. See the hint in Problem 4.20 for more information.)

• From the findings in Problem 4.18 with $u = x^2 - y^2$ and $v = 2xy$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left(\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \right)^{-1} = \left(\begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \right)^{-1} = \frac{1}{4(x^2+y^2)},$$

which makes the integral

$$\frac{1}{4}\int_0^\infty e^{-v} dv \int_{-\infty}^\infty \frac{du}{1+u^2} = \frac{1}{4}\cdot 1 \cdot \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sec^2\theta d\theta}{1+\tan^2\theta} = \frac{\pi}{4}.$$

5. Boas Chapter 5, Problem 5.1

• From Eq.(5.2) with $\phi(x,y,z) = x - 2y + 5z - 13 = 0$,

$$A = \iint \sec\gamma\, dxdy = \iint dxdy \cdot \frac{|\nabla\phi|}{|\partial\phi/\partial z|} = \int_0^3 dr \int_0^{2\pi} rd\theta \cdot \frac{|(1,-2,5)|}{5}$$
$$= 2\pi \int_0^2 rdr \cdot \frac{\sqrt{30}}{5} = \frac{9\sqrt{30}}{5}\pi.$$

6. Boas Chapter 6, Problem 3.14

(Note: For Problem 3.14, tackle the problem in two different ways, first by writing in ordinary vector notation, then by writing in tensor notation after reviewing the derivation of Eq.(5.12) in Boas Chapter 10, Section 5.)

• Using Thm.(3.9), $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) = [(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}] + [(\mathbf{B}\cdot\mathbf{A})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{A}] + [(\mathbf{C}\cdot\mathbf{B})\mathbf{A} - (\mathbf{C}\cdot\mathbf{A})\mathbf{B}] = 0$.

• Alternatively, writing in tensor notation and then using Eqs. (5.8), (5.11) and (5.12) of Boas Chapter 10, $[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})]_n = \epsilon_{nip}A_i(\mathbf{B}\times\mathbf{C})_p = \epsilon_{nip}A_i\epsilon_{pjk}B_jC_k = \epsilon_{pni}\epsilon_{pjk}A_iB_jC_k = (\delta_{nj}\delta_{ik} -$

$\delta_{nk}\delta_{ij})A_iB_jC_k = B_n(A_iC_i) - C_n(A_iB_i)$. Therefore, $[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})+\mathbf{B}\times(\mathbf{C}\times\mathbf{A})+\mathbf{C}\times(\mathbf{A}\times\mathbf{B})]_n = B_n(A_iC_i) - C_n(A_iB_i) + C_n(B_iA_i) - A_n(B_iC_i) + A_n(C_iB_i) - B_n(C_iA_i) = 0$.

7. Boas Chapter 6, Problem 6.16

• Using Eq.(9.6) of Chapter 4 and Eq.(6.2) of Chapter 6, we write $F = \frac{d\phi}{ds} + \lambda(a^2+b^2+c^2) = \left(\frac{\partial\phi}{\partial x}a + \frac{\partial\phi}{\partial y}b + \frac{\partial\phi}{\partial z}c\right) + \lambda(a^2+b^2+c^2)$. Then, $\frac{\partial F}{\partial a} = \frac{\partial\phi}{\partial x} - 2\lambda a = 0$, $\frac{\partial F}{\partial b} = \frac{\partial\phi}{\partial y} - 2\lambda b = 0$, $\frac{\partial F}{\partial c} = \frac{\partial\phi}{\partial z} - 2\lambda c = 0$, and $a^2+b^2+c^2 = 1$. Combined, $\lambda = \pm\frac{|\nabla\phi|}{2}$ and $\mathbf{u} = (a,b,c) = \frac{\nabla\phi}{2\lambda} = \pm\frac{\nabla\phi}{|\nabla\phi|}$. With Eq.(6.4), you get the extremum values of $\frac{d\phi}{ds}$ as $\nabla\phi\cdot\mathbf{u} = \pm\frac{(\nabla\phi)^2}{|\nabla\phi|} = \pm|\nabla\phi|$.

8. Boas Chapter 6, Problem 9.4

• By setting $P = e^x\cos y$ and $Q = -e^x\sin y$ and applying Eq.(9.7) of Chapter 6,

$$\begin{aligned}\iint_{BADB}\left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right)dxdy = 0 = \oint_{BADB} Pdx + Qdy &= \left(\int_{BA} Pdx + Qdy\right) + \left(\int_{ADB} Pdx + Qdy\right)\\ &= \left(\int_{-\ln 2}^{\ln 2} Pdx + Qdy\right) + \left(\int_C Pdx + Qdy\right)\\ &= \left(\int_{-\ln 2}^{\ln 2} e^x dx\right) + \left(\int_C Pdx + Qdy\right)\\ &= \frac{3}{2} + \left(\int_C Pdx + Qdy\right).\end{aligned}$$

9. In the class we discussed the Legendre transformation as one of the examples of a simple change of variables that is found to be useful in classical mechanics and thermodynamics.

(a) Review the general discussion of a Legendre transformation in Boas Chapter 4, Section 11 by following the procedure step by step from Eq.(11.21) to Eq.(11.27).

(b) In one practical example in classical mechanics, given $L(x,\dot{x})$ with $dL = \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial \dot{x}}d\dot{x} = \frac{\partial L}{\partial x}dx + pd\dot{x}$ (note $p \equiv \frac{\partial L}{\partial \dot{x}}$), one can find $H(x,p)$ so that $dH = \frac{\partial L}{\partial x}dx - \dot{x}dp$. Find a Legendre transformation that gives $H(x,p)$. Discuss the meaning of the two functions, L and H, by identifying them as Lagrangian and Hamiltonian, respectively.

(c) In another example in thermodynamics which we briefly examined but left for your exercise, given $U(S,V)$ with $dU = \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial V}dV = TdS - PdV$ (note $T \equiv \left(\frac{\partial S}{\partial U}\right)^{-1}$ and $P \equiv -\frac{\partial U}{\partial V}$), one can find $H(S,P)$ so that $dH = TdS + VdP$. Find a Legendre transformation that gives $H(S,P)$. Discuss the meaning of the two functions, U and H, by identifying them as (internal) energy and enthalpy, respectively.

(d) Given $dU = TdS - pdV$, perform a different Legendre transformation to find another useful function related to energy, $F(T,V)$, known as the Helmholtz free energy. Finally, by performing Legendre transformations on both terms in dU, find yet another function related to energy, $G(T,P)$, known as the Gibbs free energy.

(Note: You may want to briefly review the textbooks in classical mechanics such as Marion, and in statistical mechanics such as Schroeder or Reif. For (b) and (c), you are simply asked to come up with 2-3 sentences about how the these quantities are used in the respective textbooks, without going into all the mechanical or thermodynamical details. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

- (b) One can find $H(x,p) = -L + p\dot{x}$ as in Marion Eq.(7.155) with $p \equiv \frac{\partial L}{\partial \dot{x}}$ in Marion Eq.(7.151).
- (c) $H(S,P) = U + PV$ in Schroeder Eq.(1.51) with $T \equiv \left(\frac{\partial S}{\partial U}\right)^{-1}$ in Schroeder Eq.(3.4).
- (d) $F(T,V) = U - TS$ and $G(T,P) = U - TS + PV$ in Schroeder Eqs.(5.2)-(5.3).

10. (a) Review Boas Chapter 6, Example 7.2, thus prove Eq.(7.6) or Eq.(f) in p.339. You are also asked to prove the identity by explicitly working with the components, i.e.,

$$\nabla \cdot (\phi \mathbf{V}) = \frac{\partial(\phi V_x)}{\partial x} + \frac{\partial(\phi V_y)}{\partial y} + \frac{\partial(\phi V_z)}{\partial z} = ...$$

(b) Review how Gauss's law $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ is acquired from Coulomb's law in Boas Chapter 6, Section 10, where $\mathbf{E}$ is the electric field, ρ is the volume charge density, and ϵ_0 is the permittivity of free space.

(c) Prove that the energy of a continuous charge distribution is given by

$$\frac{1}{2}\int \rho V d\tau = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2 d\tau$$

for integration over all space, where V is the electrostatic potential. You are first asked to briefly discuss how one came up with the term $\frac{1}{2}\int \rho V d\tau$. To prove the equality, you may assume that V vanishes at large distance r at least as fast as r^{-1}.

(Note: The exercises here should sound familiar to most of you as you have begun to study and explore electromagnetism. If not, you may want to briefly review the classic textbooks in electromagnetism such as Griffiths or Jackson.)

- (a) Another way to prove Eq.(7.6) can be found in Section 1.2.6 of Griffiths (4th ed.).
- (c) One can think of $\frac{1}{2}\int \rho V d\tau$ as a continuum approximation of $\frac{1}{2}\sum_i q_i V(\mathbf{r}_i)$. See Section 2.4.2 of Griffiths. Using (a) and (b), one gets

$$W = \frac{1}{2}\int \rho V d\tau = \frac{\epsilon_0}{2}\int V(\nabla \cdot \mathbf{E}) d\tau = \frac{\epsilon_0}{2}\int \left[-\mathbf{E}\cdot(\nabla V) + \nabla\cdot(V\mathbf{E})\right] d\tau$$
$$= \frac{\epsilon_0}{2}\left(\int |\mathbf{E}|^2 d\tau + \oint_{\partial\tau} (V\mathbf{E})\cdot \mathbf{n} d\sigma\right).$$

The second integral vanishes as the integration is carried out over all space — since it behaves like $(r^{-1}r^{-2})\cdot r^2 = r^{-1}$ which tends to 0 as r approaches ∞.