Mathematical Physics I (Fall 2022): Homework #2 Solution

Due Sep. 30, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

1. Boas Chapter 3, Problem 5.42

(Note: For Problem 5.42, review the Examples in Boas Chapter 3, Section 5 in case you have not done it yet. Find the most relevant example and use the method depicted in there.)

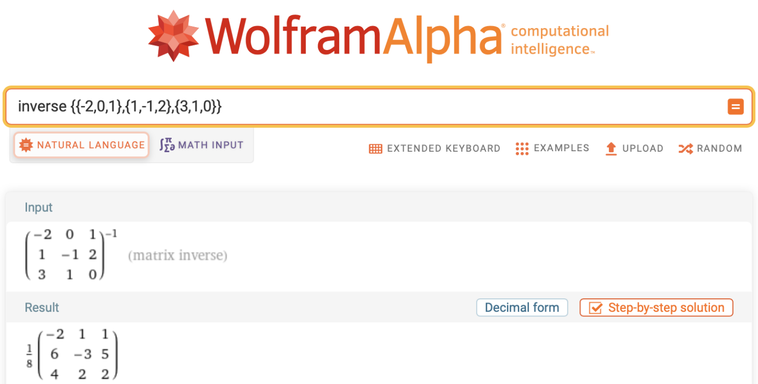
• The lines we consider are $\mathbf{r} = (\mathbf{i}+2\mathbf{j}-\mathbf{k})t = \mathbf{A}t$ and $\mathbf{r} = (\mathbf{i}+\mathbf{j}+\mathbf{k})+(\mathbf{i}+2\mathbf{j}+3\mathbf{k})t = (\mathbf{i}+\mathbf{j}+\mathbf{k})+\mathbf{B}t$. To determine the distance between the two line, let P and Q be a point on each of the lines, respectively — e.g., $P(0,0,0)$ and $Q(1,1,1)$ — and $\mathbf{n}$ be the unit vector perpendicular to both lines, $\mathbf{n} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|} = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$. Then, from Figure 5.8 and the accompanying equations, the distance between the two line is $|\overrightarrow{PQ}\cdot\mathbf{n}| = |(1,1,1)\cdot\frac{1}{\sqrt{5}}(2,-1,0)| = \frac{1}{\sqrt{5}}$.

2. Boas Chapter 3, Problem 6.16

(Note: For Problem 6.16, find the inverse with two different methods — using Eq.(6.13) of Boas Chapter 3 and using the *Gauss-Jordan matrix inversion procedure*. Compare your results with the one found with a computer.)

• From Eq.(6.13),

$$M = \begin{pmatrix} -2 & 0 & 1 \\ 1 & -1 & 2 \\ 3 & 1 & 0 \end{pmatrix} \rightarrow \det(M) = 8 \text{ and } C = \begin{pmatrix} -2 & 6 & 4 \\ 1 & -3 & 2 \\ 1 & 5 & 2 \end{pmatrix} \rightarrow M^{-1} = \frac{1}{8}\begin{pmatrix} -2 & 1 & 1 \\ 6 & -3 & 5 \\ 4 & 2 & 2 \end{pmatrix}.$$



3. Boas Chapter 3, Problem 7.35

• One may think of the given matrix as

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = BC$$

C is a reflection matrix ($\det C = -1$) through the x-y plane — which acts first on a vector $\mathbf{r}$ when you perform a transformation $A\mathbf{r} = BC\mathbf{r}$. One can also realize that the next transformation B is a 90° rotation about the z-axis.

4. Boas Chapter 3, Problem 8.16

(Note: In Problem 8.16, there is a typo that is not yet included in the errata list collected by Harold Boas. See if you can find it.)

• (b) Using the Wronskian in Eq.(8.5), for $\{x^3, |x^3|\}$ you find

$$W = \begin{vmatrix} x^3 & |x^3| \\ 3x^2 & \frac{|x|}{x} \cdot 3x^2 \end{vmatrix} = \begin{vmatrix} x^3 & x^2|x| \\ 3x^2 & 3x|x| \end{vmatrix} = 3x^4|x| - 3x^4|x| = 0,$$

yet the functions are *not* linearly dependent on $(-1, 0)$.

5. Boas Chapter 3, Problem 9.17

• (c) $A = A^\dagger$ and $B = B^\dagger$ $\rightarrow$ from a complex counterpart of Eq.(9.11), $(AB)^\dagger = B^\dagger A^\dagger = BA = AB$ holds (i.e., AB is Hermitian) if and only if $[A, B] = 0$.

• (d) $A^{-1} = A^\dagger$ and $B^{-1} = B^\dagger$ $\rightarrow$ from Eq.(9.12), and from a complex counterpart of Eq.(9.11), $(AB)^{-1} = B^{-1}A^{-1} = B^\dagger A^\dagger = (AB)^\dagger$ holds (i.e., AB is unitary).

6. Boas Chapter 3, Problem 11.32

• From the characteristic equation, two eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 7$ appear that correspond to eigenvectors $\mathbf{r}_1 = \frac{1}{\sqrt{5}}\binom{1}{2}$ and $\mathbf{r}_2 = \frac{1}{\sqrt{5}}\binom{2}{-1}$, respectively. Then it is straightforward to show that

$$C^{-1}MC = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

which is a deformation making the x-coordinate twice as large and the y-coordinate seven times as large (relative to the new axes).

7. Boas Chapter 3, Problem 11.60

(Note: For Problem 11.60, you will need to first prove then utilize the findings in Problem 11.57, or Eq.(11.36).)

• With $C = \frac{1}{\sqrt{5}}\left(\begin{smallmatrix}1 & -2\\ 2 & 1\end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix}1 & 0\\ 0 & 6\end{smallmatrix}\right) = \left(\begin{smallmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{smallmatrix}\right)$ from Eq.(11.10),

$$\begin{aligned} C^{-1}(M^2 - 7M + 6I)C &= C^{-1}M^2C - 7C^{-1}MC + 6C^{-1}C \\ &= D^2 - 7D + 6I = \begin{pmatrix}\lambda_1^2 - 7\lambda_1 + 6 & 0 \\ 0 & \lambda_2^2 - 7\lambda_2 + 6\end{pmatrix} = 0, \end{aligned}$$

which indicates that $M^2 - 7M + 6I = C(D^2 - 7D + 6I)C^{-1} = 0$.

8. Boas Chapter 3, Problem 12.9 (for 12.4)

(Note: For Problem 12.9, consider only Problem 12.4.)

• As in Example 1 of Boas Chapter 3, Section 12,

$$(x, y)\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 8 \;\rightarrow\; \begin{vmatrix}3-\lambda & 4\\ 4 & -3-\lambda\end{vmatrix} = 0 = (\lambda - 5)(\lambda + 5),$$

from which $\lambda_1 = 5$ and $\lambda_2 = -5$ are acquired. Therefore, the new conic equation relative to the principal axes becomes

$$(x', y')\begin{pmatrix}5 & 0\\ 0 & -5\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} = 8 \;\rightarrow\; \frac{x'^2}{(8/5)} - \frac{y'^2}{(8/5)} = 1$$

which represents a hyperbola. The corresponding eigenvectors $\mathbf{r}_1 = \frac{1}{\sqrt{5}}\left(\begin{smallmatrix}2\\ 1\end{smallmatrix}\right)$ and $\mathbf{r}_2 = \frac{1}{\sqrt{5}}\left(\begin{smallmatrix}1\\ -2\end{smallmatrix}\right)$ provides us with the rotation matrix C. Then it is straightforward to show that

$$C^{-1}MC = \frac{1}{5}\begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix}\begin{pmatrix}3 & 4\\ 4 & -3\end{pmatrix}\begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}.$$

9. In this problem we consider the spin matrices in quantum mechanics that describe particles of various spins in three dimensions.

(a) First, work out Problem 6.6 in Boas Chapter 3. Here, for the Pauli spin matrices introduced to describe particles of spin 1/2,

$$A = \sigma_x = \sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad B = \sigma_y = \sigma_2 = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad C = \sigma_z = \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

you will first need to show that $\sigma_j\sigma_k = \delta_{jk}I_2 + i\sum_l \epsilon_{jkl}\sigma_l$, where δ_{jk} and ϵ_{jkl} are defined in Eq.(9.4) of Boas Chapter 3 and in Eq.(5.3) of Chapter 10, respectively, and I_n is the $n \times n$ unit matrix. Then, it naturally follows that $[\sigma_j, \sigma_k] \equiv \sigma_j\sigma_k - \sigma_k\sigma_j = 2i\sum_l \epsilon_{jkl}\sigma_l$, which is called the *fundamental commutation relation* for angular momentum matrices (or $[\sigma_j, \sigma_k] = 2i\sigma_l$ if $j, k, l = 1, 2, 3$ or a cyclic permutation thereof).

(b) Briefly discuss how these spin matrices are introduced and used in quantum mechanics. Prove that σ_x, σ_y and σ_z are both Hermitian and unitary. Show also that $\sigma^2 \equiv \sum_j \sigma_j^2 = 3I_2$.

(c) Now, using the 3×3 spin matrices that can describe particles of spin 1,

$$M_x = M_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad M_y = M_2 = \frac{i}{\sqrt{2}}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}, \quad M_z = M_3 = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix},$$

show that $[M_j, M_k] = i \sum_l \epsilon_{jkl} M_l$ and $M^2 \equiv \sum_j M_j^2 = 2I_3$.

(d) Finally, using the 4×4 spin matrices that can describe particles of spin 3/2,

$$M_x = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad M_y = \frac{i}{2}\begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad M_z = \frac{1}{2}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

show that $[M_j, M_k] = i \sum_l \epsilon_{jkl} M_l$. Briefly explain only in words why we need larger-size matrices to describe particles with a larger spin.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. For (b), you are simply asked to come up with 2-3 sentences about how the matrices are used, without diving into laborious quantum mechanical derivations. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

- (a) By combining $\sigma_j \sigma_k = i\sigma_l$ (if $j, k, l = 1, 2, 3$ or a cyclic permutation thereof) and $(\sigma_j)^2 = I_2$ (where I_2 is the 2×2 unit matrix), we can easily show $\sigma_j \sigma_k = \delta_{jk} I_2 + i \sum_l \epsilon_{jkl} \sigma_l$.

- (c) You may compare your answer from direct matrix multiplication with a computer solution.

SOURCE CODE

```
import numpy as np

A=np.multiply(1./np.sqrt(2), [[0,1,0],[1,0,1],[0,1,0]])
B=np.multiply(1./np.sqrt(2), [[0,-1j,0],[1j,0,-1j],[0,1j,0]])
C=[[1,0,0],[0,0,0],[0,0,-1]]

print("...Checking commutation relations...!")
print(np.dot(A,B)-np.dot(B,A))
print(np.multiply(1j, C))
print(np.dot(B,C)-np.dot(C,B))
print(np.multiply(1j, A))
print(np.dot(C,A)-np.dot(A,C))
print(np.multiply(1j, B))
print("")
print("...Checking M^2...!")
print(np.dot(A,A)+np.dot(B,B)+np.dot(C,C))
```

RESULTS

```
...Checking commutation relations...!
[[ 0.+1.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.-1.j]]
[[ 0.+1.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j -0.-1.j]]
[[ 0.+0.j          0.+0.70710678j  0.+0.j        ]
 [ 0.+0.70710678j  0.+0.j          0.+0.70710678j]
 [ 0.+0.j          0.+0.70710678j  0.+0.j        ]]
[[ 0.+0.j          0.+0.70710678j  0.+0.j        ]
 [ 0.+0.70710678j  0.+0.j          0.+0.70710678j]
 [ 0.+0.j          0.+0.70710678j  0.+0.j        ]]
[[ 0.          0.70710678  0.        ]
 [-0.70710678  0.          0.70710678]
 [ 0.         -0.70710678  0.        ]]
[[ 0.00000000+0.j  0.70710678+0.j  0.00000000+0.j]
 [-0.70710678+0.j  0.00000000+0.j  0.70710678+0.j]
 [ 0.00000000+0.j -0.70710678+0.j  0.00000000+0.j]]

...Checking M^2...!
[[ 2.+0.j  0.+0.j  0.+0.j]
 [ 0.+0.j  2.+0.j  0.+0.j]
 [ 0.+0.j  0.+0.j  2.+0.j]]
```

10. In the class we discussed the Gram-Schmidt orthonormalization process for a linear vector space and for a general vector space.

(a) In one example we briefly examined but left for your exercise (Example 14.6 of Boas Chapter 3), with an inner product defined as

$$\langle f|g\rangle = \int_{-1}^{1} f^*(x)g(x)dx,$$

one can construct a set of orthonormal polynomials P_i that satisfy the orthonormality condition on the interval $-1 \le x \le 1$,

$$\int_{-1}^{1} P_m(x) P_n(x) dx = \delta_{mn}$$

(see also Eq.(8.4) of Chapter 12, but note a different normalization factor). We later identify this set of functions as *Legendre polynomials.* Starting from Eq.(14.10), follow the procedure step by step and find for yourself the first four members of P_i.

(b) In a similar manner, with an inner product defined as

$$\langle f|g\rangle = \int_0^{\infty} f^*(x) g(x) e^{-x} dx,$$

find the first three members in the set of orthonormal polynomials L_i that satisfy the orthonormality condition on the interval $0 \le x \le \infty$,

$$\int_0^{\infty} L_m(x) L_n(x) e^{-x} dx = \delta_{mn}$$

(see also Eq.(22.22) of Chapter 12). We call this set of functions as *Laguerre polynomials.*

(c) Discuss briefly where the (associated) Legendre polynomials and the (associated) Laguerre polynomials appear in quantum mechanics or in other physics research.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. Once again, you are simply asked to come up with 2-3 sentences about how these polynomials are used, not the detailed physical or mathematical discussions. *Astronomy majors* are encouraged to look for the usage of Legendre polynomials in astrophysics research.)

- (b-1) Following the notation in Example 14.6 of Boas Chapter 3, but with $L_i = e_i$, first we find $L_0 = p_0 = f_0 = 1$ as $||p_0||^2 = \int_0^{\infty} e^{-x} dx = 1$.
- (b-2) Then $p_1 = f_1 - L_0 \int_0^{\infty} L_0 f_1 e^{-x} dx = x - 1 \cdot \int_0^{\infty} x e^{-x} dx = x - 1 \ \rightarrow$ therefore $L_1 = \pm(x-1)$ since $||p_1||^2 = \int_0^{\infty} (x-1)^2 e^{-x} dx = 1$.
- (b-3) The list of the first few normalized Laguerre polynomials are given in Eq.(22.20) of Chapter 12, or in p.804 (the answer to Problem 22.13 of Chapter 12).