Mathematical Physics I (Fall 2022): Homework #1 Solution

Due Sep. 16, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

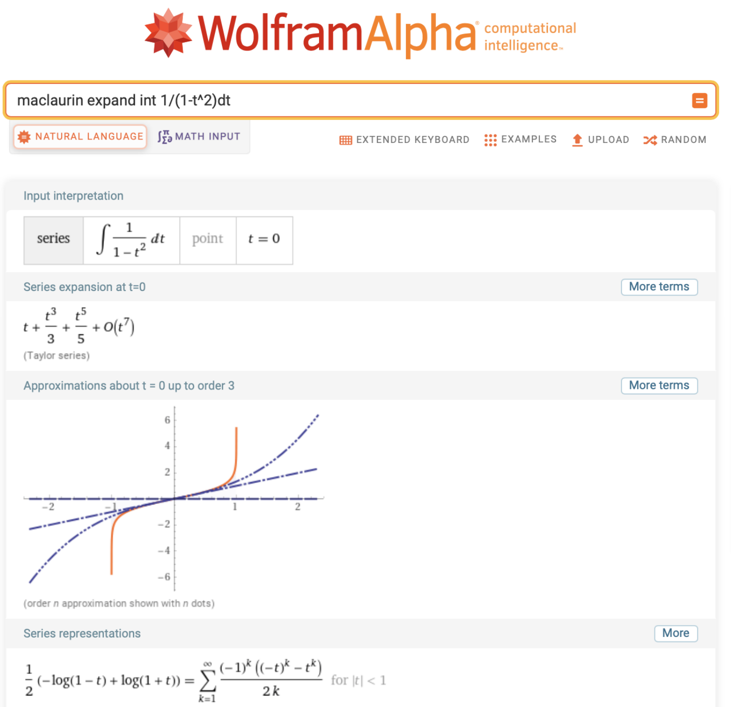
1. Boas Chapter 1, Problem 9.22(a)(b)

• The series converges in (a) and diverges in (b) according to the comparison (or integral) test.

2. Boas Chapter 1, Problem 13.14

(Note: For Problem 13.14, first prove the equality given, for which you may need to utilize $\frac{1}{1-t^2} = \frac{1}{2}\left(\frac{1}{1+t} + \frac{1}{1-t}\right)$. Read the instruction in the textbook carefully; among others, you have been asked to compare your result with a computer solution as well.)

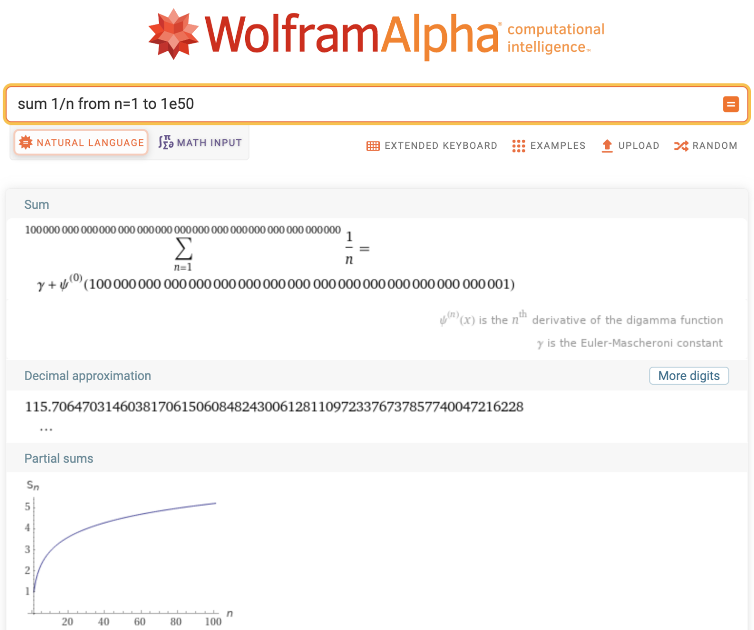
• From Eq.(13.5), you can get the binomial series expansion $(1-t^2)^{-1} \simeq 1-(-1)t^2 + \frac{(-1)(-2)}{2!}t^4$. Integrating both sides, you find $\int_0^x (1-t^2)^{-1} dt \simeq x + \frac{1}{3}x^3 + \frac{1}{5}x^5$. The general terms of the Maclaurin series of the given integral are found as $\int_0^x (1-t^2)^{-1} dt = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$.



3. Boas Chapter 1, Problem 16.1

(Note: For Problem 16.1, it is helpful to draw a diagram that considers the balance between forces and torques, a so-called *extended free-body diagram.*)

• You may find it interesting to read an article about the stacking problem such as Hall, J., 2005, "Fun with stacking block", American Journal of Physics, 73, 1107.



4. Boas Chapter 2, Problem 9.23

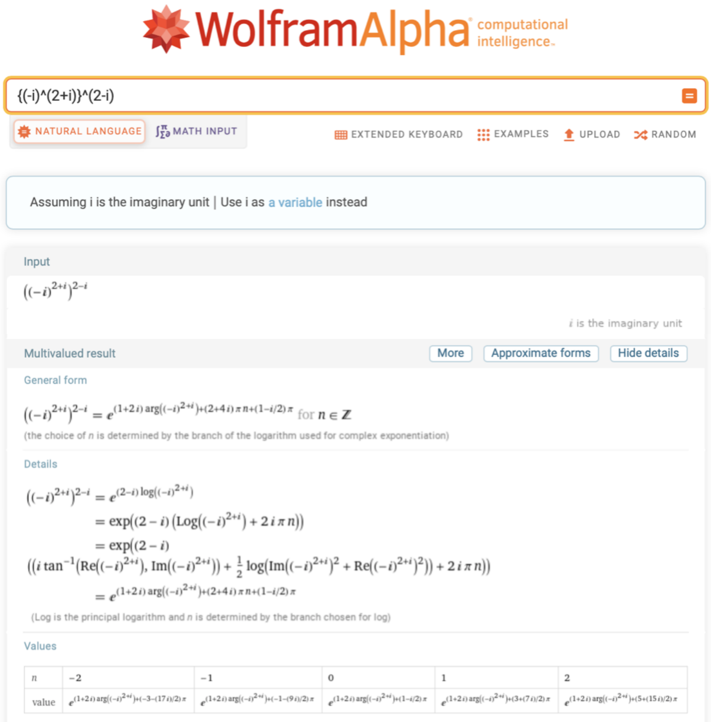
• $\frac{(1+i)^{48}}{(\sqrt{3}-i)^{25}} = \frac{(\sqrt{2}e^{i\frac{\pi}{4}})^{48}}{(2e^{-i\frac{\pi}{6}})^{25}} = \frac{(e^{i\frac{\pi}{6}})}{2} = \frac{\sqrt{3}+i}{4}$.

5. Boas Chapter 2, Problem 14.24

(Note: For Problem 14.24, compare your result with a computer solution. Resolve or explain any disagreement in the two results.)

• (a-1) Using the definition of complex powers in Eq.(14.1), $\left[(-i)^{2+i}\right]^{2-i} = \left[e^{(2+i)\ln(-i)}\right]^{2-i} = \left[e^{(2+i)i\left(\frac{3\pi}{2}+2n\pi\right)}\right]^{2-i} = \left[e^{2i\left(\frac{3\pi}{2}+2n\pi\right)} \cdot e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]^{2-i} = \left[-e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]^{2-i} = e^{(2-i)\ln\left[-e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]} = e^{(2-i)\left[i(\pi+2n\pi)+\ln\left[e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]\right]} = e^{2\ln\left[e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]+(\pi+2n\pi)} \cdot e^{i\left(-\ln\left[e^{-\left(\frac{3\pi}{2}+2n\pi\right)}\right]+2(\pi+2n\pi)\right)}$.

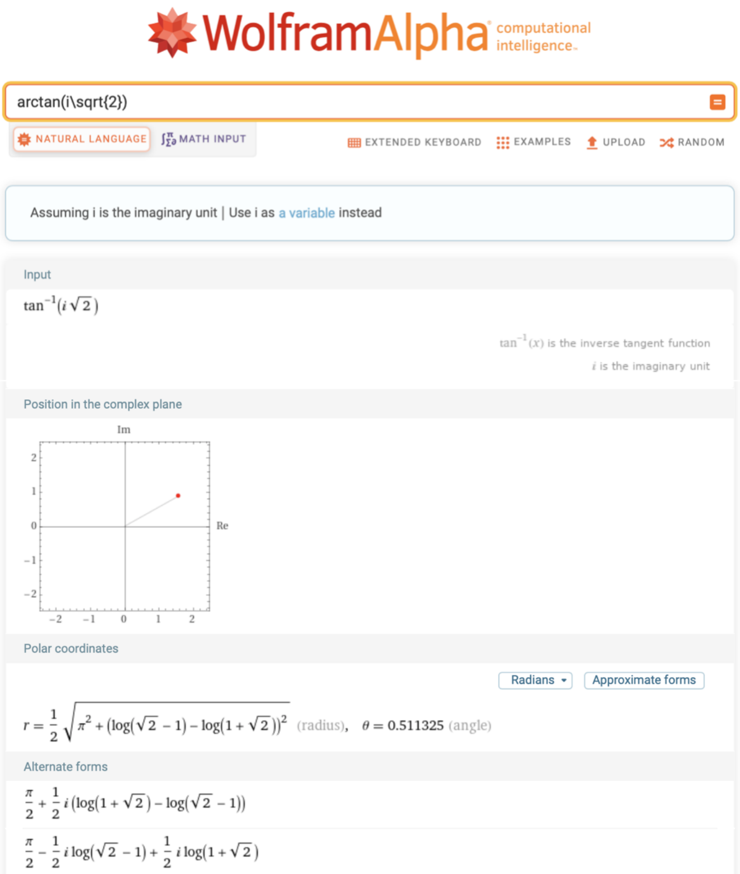
• (a-2) Meanwhile, $(-i)^{(2+i)(2-i)} = -i$, which can be confirmed with the computer solution below. Note that, from our analytic calculations and computer solutions, your answer in (a-2) is a special case in your answer in (a-1).



6. Boas Chapter 2, Problem 15.7

(Note: For Problem 15.7, read the instruction in the textbook carefully; that is, you have been asked to compare your result with a computer solution as well. Resolve or explain any disagreement in the two results.)

- With $u \equiv e^{iz}$, you get $\tan z = i\sqrt{2} = \frac{-i(e^{iz}-e^{-iz})}{e^{iz}+e^{-iz}} = \frac{-i(u-u^{-1})}{u+u^{-1}} \;\to\; u^2 = \frac{1-\sqrt{2}}{1+\sqrt{2}} = -(\sqrt{2}-1)^2$ $\to\; iz = \ln u = \ln\left[\pm i(\sqrt{2}-1)\right] = \ln\left[(\sqrt{2}-1)e^{i(\frac{\pi}{2}+n\pi)}\right] = \mathrm{Ln}\,(\sqrt{2}-1) + i(\frac{\pi}{2}+n\pi)$.



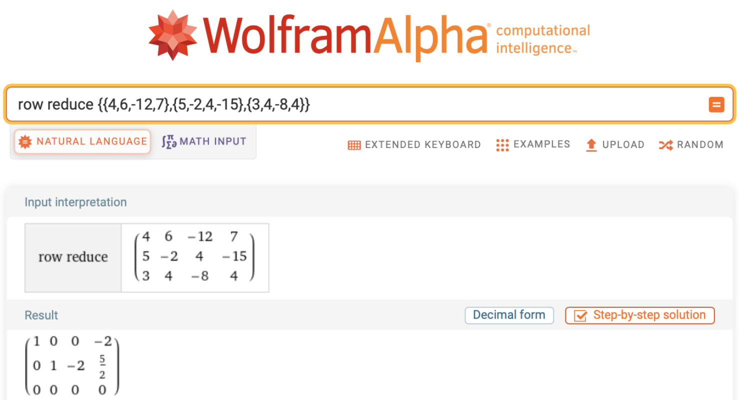
7. Boas Chapter 2, Problem 17.19

- $\arctan z = \omega \;\rightarrow$ With $u \equiv e^{i\omega}$, you find $z = \tan\omega = \frac{e^{i\omega} - e^{i\omega}}{i(e^{i\omega}+e^{i\omega})} = \frac{u-u^{-1}}{i(u+u^{-1})} \;\rightarrow\; u^2 = \frac{1+iz}{1-iz} \;\rightarrow$
$\omega = \frac{1}{i}\ln u = \frac{1}{i}\ln\left(\frac{1+iz}{1-iz}\right)^{\frac{1}{2}} = \arctan z.$

8. Boas Chapter 3, Problem 2.13

(Note: For Problem 2.13, again, read the instruction in the textbook carefully.)

- The rank of the augmented matrix is 2 while there are 3 unknowns.



9. In the quantum statistical distribution of bosons with unspecified total number of particles (so-called Bose-Einstein statistics of quantized oscillators), the average energy of the system is found to be

$$\bar{\epsilon} = \frac{\sum\limits_{n=1}^{\infty} n\epsilon_0 e^{-\frac{n\epsilon_0}{k_{\mathrm{B}}T}}}{\sum\limits_{n=0}^{\infty} e^{-\frac{n\epsilon_0}{k_{\mathrm{B}}T}}}, \tag{1}$$

where ϵ_0 is a fixed energy, k_B is the Boltzmann constant, and T is the temperature. (a) After briefly discussing how this formula is acquired, show that the ratio becomes

$$\bar{\epsilon} = \frac{\epsilon_0}{e^{\frac{\epsilon_0}{k_{\mathrm{B}}T}} - 1}$$

by first identifying Eq.(1)'s denominator as a binomial expansion of $\frac{1}{1-x}$ with $x = e^{-\frac{\epsilon_0}{k_{\mathrm{B}}T}}$, and its numerator as a constant times $\frac{x}{(1-x)^2}$. (b) Using a power series expansion, show that $\bar{\epsilon}$ reduces to $k_{\mathrm{B}}T$ in the classical limit of $k_{\mathrm{B}}T \gg \epsilon_0$.

(Note: You may want to briefly review the classic textbooks in statistical mechanics such as Schroeder or Reif. To acquire a full credit, you are asked to come up with a short paragraph about what Eq.(1) means, without having to laboriously derive the equation in great detail. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

- (a) By differentiating $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} x^n$, you find $\frac{x}{(1-x)^2} = \sum\limits_{n=1}^{\infty} nx^n$, which leads to

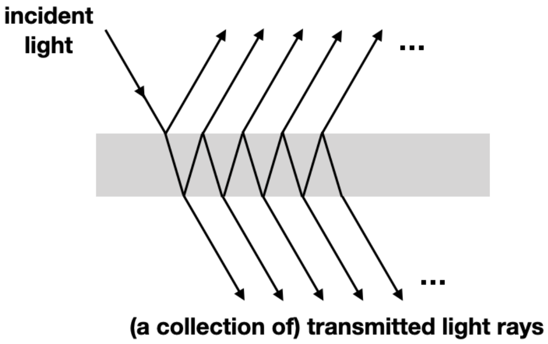
$$\bar{\epsilon} = \frac{\sum\limits_{n=1}^{\infty} \epsilon_0 n x^n}{\sum\limits_{n=0}^{\infty} x^n} = \frac{\epsilon_0 x}{1-x} = \frac{\epsilon_0}{x^{-1}-1}.$$

- (b) With $y = \frac{\epsilon_0}{k_{\mathrm{B}}T} \ll 1$, you get $\bar{\epsilon} = \frac{\epsilon_0}{e^y - 1} \simeq \frac{\epsilon_0}{1+y-1} = k_{\mathrm{B}}T$.

10. Let us consider light transmission through a thin film in optics. Explain why the following expression needs to be computed when evaluating the intensity of light transmitted through a thin film after multiple reflections at the surfaces of the film:

$$I = \left| \sum_{n=0}^{\infty} r^{2n} e^{in\theta} \right|^2 = \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta \right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta \right)^2 = \frac{1}{1 - 2r^2 \cos\theta + r^4} \tag{2}$$

where r is the fraction of light reflected each time, and θ is the phase difference between successive transmitted terms. Prove each of the equalities above assuming $|r| < 1$.



(Note: You may want to review the freshman physics textbooks such as Halliday & Resnick. To acquire a full credit, you need to come up with a short paragraph about how Eq.(2) is used when describing the sum of all the light intensities of the transmitted rays seen above.)

- $$I = \left|\sum_{n=0}^{\infty} r^{2n} e^{in\theta}\right|^2 = \left|\sum_{n=0}^{\infty} r^{2n} \cos n\theta + i\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right|^2 = \left(\sum_{n=0}^{\infty} r^{2n} \cos n\theta\right)^2 + \left(\sum_{n=0}^{\infty} r^{2n} \sin n\theta\right)^2$$
$$= \left|\frac{1}{1 - r^2 e^{i\theta}}\right|^2 = \left|\frac{1}{1 - r^2 \cos\theta + i r^2 \sin\theta}\right|^2 = \frac{1}{(1 - r^2\cos\theta)^2 + r^4 \sin^2\theta}\,.$$