Mathematical Physics I (Fall 2022): Homework #6

Due Dec. 2, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.

1.-2. Boas Chapter 8, Problems 11.7, 12.2

(Note: For Problem 11.7, you will first want to review Problem 11.6(b) and (c), and Example 10.1. You may assume $y_0 = y_0' = 0$ as in Problem 11.6(a). You may opt to utilize the *translation theorems* of the Laplace transform described in Problems 8.19 to 8.27 of Boas Chapter 8. For Problem 12.2, note that we assume $t > 0$ and $y_0 = y_0' = 0$ at $t = 0$; see the text right before Eq.(12.6) in Boas Chapter 8.)

3.-5. Boas Chapter 13, Problems 2.3, 3.3, 4.2

(Note: For Problem 2.3 to 4.2, carefully read the instruction at the beginning of the Problems; that is, you are asked to make a computer plot using the first several terms of your answer. For Problem 3.3, you may want to review Example 2 of Boas Chapter 13, Section 3.)

6.-8. Boas Chapter 9, Problems 2.3, 4.4, 8.5

(Note: For Problem 2.3, you may want to utilize the indefinite integral found in Problem 13.19 of Boas Chapter 1 (— along with the formula in Problem 17.20 of Chapter 2), or in extensive references like Zwillinger (Sections 5.4.1 and 6.11.1, 33rd ed.). For Problem 4.4, as you evaluate the integral for T you may utilize the indefinite integral $\int \frac{dx}{\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{-a}}\sin^{-1}(\frac{2ax+b}{\sqrt{b^2-4ac}})$ (if $a < 0$ and $b^2 > 4ac$) found in common integral tables like Appendix E of Thornton & Marion, or in extensive references like Zwillinger (Section 5.4.13, 33rd ed.). For Problem 8.5, you are asked to change the independent variable, as in Example 3 of Boas Chapter 9, Section 3.)

9. In the class we discussed the resemblance between the heat flow equation and the Schrödinger equation, and left the latter for your exercise.

(a) Review how the *time-independent* Schrödinger equation, Eq.(3.22), and the time equation, Eq.(3.21), are acquired from the Schrödinger equation, Eq.(1.6), in Boas Chapter 13.

(b) Then, review the 1-dimensional "particle in a box" problem in Example 3 of Boas Chapter 13, Section 3. That is, starting from Eq.(3.23) with $V = 0$ on $(0, l)$ and boundary conditions $\Psi(0, t) = 0$ and $\Psi(l, t) = 0$, show how you can reach $\Psi(x, t)$ in Eq.(3.26) with $E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{l}\right)^2$.

(c) Now we consider an initial condition. Verify explicitly that an initial condition $\Psi(x, 0) = \frac{100}{l}x$ yields $\Psi(x, t)$ in Eq.(3.27). Repeat the exercise and find $\Psi(x, t)$ with a different initial condition $\Psi(x, 0) = 1$ on $(0, l)$.

(d) Here, we extend the problem in (b) to three dimensions. Show how your answer in (b) changes if you tackle the 3-dimensional "particle in a box" problem (e.g., a box with planar faces at $x = 0$, $x = l_x$, $y = 0$, $y = l_y$, $z = 0$, $z = l_z$). You are asked to explicitly separate the variables in Eq.(3.22) with $V = 0$, by assuming a solution of the form $\psi(x, y, z) = X(x)Y(y)Z(z)$, to reach

$$\Psi(x, y, z, t) = \sum_{n_x,\, n_y} A_{n_x n_y} \sin\frac{n_x \pi x}{l_x} \sin\frac{n_y \pi y}{l_y} \sin\frac{n_z \pi z}{l_z}\, e^{-iEt/\hbar},$$

where n_z is determined by $E = \frac{\hbar^2\pi^2}{2m}\left[\left(\frac{n_x}{l_x}\right)^2 + \left(\frac{n_y}{l_y}\right)^2 + \left(\frac{n_z}{l_z}\right)^2\right]$ for a given n_x, n_y, and E.

(e) Finally, we extend the problem in (b) to the case with a harmonic potential — i.e., a particle in a potential field $V = \frac{1}{2}kx^2$. Starting from the equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi = E\psi,$$

(i.e., Eq.(3.23) of Boas Chapter 13 with $V = \frac{1}{2}kx^2$, or Eq.(2.45) of Griffiths & Schroeter) and changing the variable with $\xi = \alpha x = \left(\frac{mk}{\hbar^2}\right)^{\frac{1}{4}} \cdot x$ and $\lambda = \frac{2E}{\hbar}\left(\frac{m}{k}\right)^{\frac{1}{2}}$, demonstrate that $\psi(\xi)$ satisfies the equation,

$$-\frac{d^2\psi}{d\xi^2} + \xi^2\psi = \lambda\psi.$$

Then, assuming a solution of the form $\psi(\xi) = y(\xi)e^{-\xi^2/2}$, verify that $y(\xi)$ satisfies

$$y'' - 2\xi y' + (\lambda - 1)y = 0,$$

which is the Hermite equation seen in Eq.(22.14) of Boas Chapter 12, Section 22 (with $\lambda = 2n + 1$), or in Eq.(2.79) of Griffiths & Schroeter. From this, prove that the possible wave functions for the harmonic oscillator potential are written as

$$\psi_n(\xi) = A_n H_n(\xi)e^{-\xi^2/2},$$

where $H_n(\xi)$ is the nth Hermite polynomial with the corresponding energy $E_n = \left(n + \frac{1}{2}\right)\hbar\left(\frac{k}{m}\right)^{\frac{1}{2}}$.

(Note: The Schrödinger equation discussed here should look familiar to most of you as you have studied the basic "particle in a box" problem in your elementary physics classes. If not, you may want to briefly review the freshman physics textbooks such as Halliday & Resnick, or the quantum mechanics textbooks such as Griffiths & Schroeter. For (e), briefly discuss how one could come up with the educated guess $\psi(\xi) = y(\xi)e^{-\xi^2/2}$. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

10. In the class we briefly discussed that the crucial properties of the Dirac delta function $\delta(x)$ may be developed as the limiting case using any of the following sequences of functions:

$$\delta_{n,1}(x) = \begin{cases} n, & \text{if } |x| < \frac{1}{2n} \\ 0, & \text{otherwise,} \end{cases} \qquad \delta_{n,2}(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2},$$

$$\delta_{n,3}(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}, \qquad \delta_{n,4}(x) = \frac{n}{2\cosh^2 nx},$$

$$\delta_{n,5}(x) = \frac{\sin nx}{\pi x}, \qquad \delta_{n,6}(x) = \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{1}{2}x}, \text{ etc.}$$

In other words, we may regard $\delta(x)$ as a normalized *distribution* which is defined with

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx. \qquad (1)$$

(a) Using $\delta_{n,1}(x)$ to $\delta_{n,5}(x)$, show that

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1 \quad \text{for all } n.$$

(b) Using $\delta_{n,1}(x)$ and $\delta_{n,5}(x)$, prove

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0).$$

(c) Treating $\delta_n(x)$ and its derivative as in Eq.(1), prove one of the properties of $\delta(x)$,

$$x\delta'(x) = -\delta(x).$$

(Note: For $\delta_{n,3}(x)$ in (a), you may need to prove and use the indefinite integral $\int \frac{dx}{1+x^2} = \arctan x$. For $\delta_{n,4}(x)$, you may simply utilize the indefinite integral $\int \text{sech}^2 x\, dx = \tanh x$ found in common integral tables, or in extensive references like Zwillinger (Section 5.4.20, 33rd ed.). For $\delta_{n,5}(x)$, you may also take advantage of the definite integral $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$ found in Example 2 of Boas Chapter 7, Section 12 or in Example 4 of Chapter 14, Section 7.)