Mathematical Physics I (Fall 2022): Homework #5

Due Nov. 18, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.

1.-8. Boas Chapter 8, Problems 5.24, 6.10, 6.26, 6.34, 7.3, 7.22, 8.12, 9.16

(Note: For Problems in Sections 5 and 6, read the instruction in the textbook carefully; that is, you have been asked to find a computer solution and reconcile differences, if any. For Problem 5.24, you will first want to review Problem 5.21. You may continue to utilize computer solutions to validate your answers to problems in Sections 7 to 9.)

9. Let us apply the Laplace transform to two examples in nuclear physics.

(a) Consider a series of radioactive decays between three nuclides, with Nuclide 1 decaying into Nuclide 2, and Nuclide 2 into Nuclide 3. The concentration fo the nuclides satisfy the system of differential equations

$$\frac{dN_1}{dt} = -\lambda_1 N_1, \quad \frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2, \quad \frac{dN_3}{dt} = \lambda_2 N_2,$$

where one can see that Nuclide 3 is assumed to be stable. λ_1 and λ_2 are decay constants. Explain the meaning of each term in each equation. Then, with initial conditions $N_1(0) = N_0$, $N_2(0) = 0$, and $N_3(0) = 0$, find $N_1(t)$, $N_2(t)$, and $N_3(t)$. Tackle the problem in two different ways, first by using the methods described in Boas Chapter 8, Sections 2 and 3, then by using the Laplace transform.

(b) Now consider a different type of radioactive decay seen in a nuclear reactor. The rate of change in concentration of Nuclide 2 is now modified to

$$\frac{dN_2}{dt} = \phi(\sigma_1 N_0 - \sigma_2 N_2) - \lambda_2 N_2,$$

where one can see that the concentration of Nuclide 1 is assumed to be (approximately) constant — i.e., $N_1(t) = N_1(0) = N_0$. ϕ is the neutron flux (in $\text{cm}^{-2}\,\text{s}^{-1}$) and σ_1 and σ_2 (in cm^2) are neutron absorption cross sections. Briefly explain only in words the meaning of each term in the equation. Then, with an initial condition $N_2(0) = 0$, find $N_2(t)$.

(Note: If you are not familiar with the topics discussed here such as radioactive decay or a nuclear reactor, you may review the freshman physics textbooks such as Halliday & Resnick. For (a), you may also want to review Example 2 in Boas Chapter 8, Section 3. For (b), find the numerical value of N_2 at $t = 1$ year, the concentration of ^{154}Eu which the original isotope ^{153}Eu is decaying into, using the following constants: $\sigma_1 = 4 \times 10^{-22}\text{cm}^2$, $\sigma_2 = 10^{-21}\text{cm}^2$, $\lambda_2 = 1.4 \times 10^{-9}\text{s}^{-1}$, $\phi = 10^9\text{cm}^{-2}\,\text{s}^{-1}$, and $N_0 = 10^{20}$. Check if the assumption that $N_1(t) = N_0$ is justified. Control rods in nuclear reactors should be made of elements capable of absorbing many neutrons without themselves decaying, such as ^{153}Eu.)

10. In the beginning of Boas Chapter 8, Section 7, the author discusses many methods of solving various types of second-order ODEs. Among them is Lagrange's *method of variation of parameters* to find a particular solution of an inhomogeneous ODE.

(a) Let us start with a homogeneous second-order linear ODE in the form of

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous functions of x. Let us assume that we know its two independent solutions, y_1 and y_2. Now, for the inhomogeneous second-order linear ODE of

$$y'' + p(x)y' + q(x)y = f(x),$$

show that a particular solution $y_p(x)$ is written as

$$y_p(x) = -y_1(x) \int \frac{y_2(x')f(x')}{W(x')} dx' + y_2(x) \int \frac{y_1(x')f(x')}{W(x')} dx'$$

where $W(x')$ is the Wronskian of y_1 and y_2, $W(y_1(x'), y_2(x'))$.

(Note: You may start with $y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$ and follow the step-by-step instruction given in Boas Chapter 8, Problem 12.14(b) that leads to the set of two conditions for c_1 and c_2: $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ and $c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = f(x)$. Notice that the first equation of this set is our "imposed" condition, while the second one is what you get if you plug $y_p' = \{c_1'(x)y_1(x) + c_2'(x)y_2(x)\} + \{c_1(x)y_1'(x) + c_2(x)y_2'(x)\} = c_1(x)y_1'(x) + c_2(x)y_2'(x)$ and the corresponding y_p'' into our ODE above. In case you wonder, no knowledge about the Green function in Section 12 is needed to tackle this problem.)

Now, utilizing the given solution of the homogeneous equation, find a solution of each of the following inhomogeneous ODEs. (More exercise problems in Chapter 8, Problems 12.15-18.)

(b) $y'' + y = \sec x$; with $y_1 = \cos x$ and $y_2 = \sin x$

(c) $(1 - x)y'' + xy' - y = (1 - x)^2$; with $y_1 = x$ and $y_2 = e^x$