Mathematical Physics I (Fall 2022): Homework #4

Due Nov. 4, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as Matlab or Mathematica.

1.-6. Boas Chapter 7, Problems 5.4, 6.12 (for Problem 5.9), 7.9, 9.20, 12.26, 13.4

(Note: For Problem 5.4, you are asked to tackle the problem in two ways — first directly find the Fourier coefficients, then verify your result using the answer given in Problem 5.3. For Problems 6.12 and 7.9, you will first have to work out Problem 5.9. You are asked to tackle Problem 5.9 in two ways — first directly find the Fourier coefficients, then verify your result using the answer given in Problem 5.7. Then for Problem 6.12 you may consider a way to automatically draw several partial sums of different n's — e.g., a simple script written in Matlab or in `python`; however, making such an automated script is not necessary for you to receive a full credit. For Problem 9.20, check out Example in Boas Chapter 7, Section 9 for a worked example. For Problem 12.26, you are asked to first work out Problem 12.15. For Problem 13.4(a), use the technique discussed in Boas Chapter 8, Section 2.)

7.-8. Boas Chapter 8, Problems 2.8, 3.14

(Note: For Problem 2.8, read the instruction in the textbook carefully; that is, you have been asked to plot a slope field with a computer, for example. Read Boas Chapter 8, Section 1 to learn about the "slope field". For Problem 3.14, again, read the instruction and the hint in the textbook carefully.)

9. Let us use the Fourier transform to appreciate the meaning of the Heisenberg uncertainty principle in quantum mechanics.

(a) Imagine an infinite wave train $\sin\omega_0 t$ clipped by shutters to maintain only N cycles of the original waveform:

$$f(t) = \begin{cases} \sin\omega_0 t, & \text{if } |\omega_0 t| < N\pi, \\ 0, & \text{if } |\omega_0 t| > N\pi. \end{cases}$$

Find the amplitude function of the Fourier (exponential) transform, $g(\omega)$. Since the prefactor may depend on the exact definition of the transform, do not worry too much about it.

(b) Find the amplitude function of the Fourier sine transform, $g_s(\omega)$. Again, since the prefactor may depend on the exact definition of the transform, do not worry too much about it.

(c) Show that, in the special case of $N = \frac{1}{2}$ and $\omega_0 = 1$, $g_s(\omega)$ becomes

$$g_{s,1}(\omega) = \sqrt{\frac{2}{\pi}}\,\frac{\omega\cos\left(\frac{\omega\pi}{2}\right)}{1-\omega^2}.$$

(d) Now consider the limit of $\omega_0 \gg 1$ and $\omega \approx \omega_0$. Show that $g_s(\omega)$ is approximately equal to

$$g_{s,2}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{\sin[(\omega_0-\omega)\frac{N\pi}{\omega_0}]}{\omega_0-\omega}.$$

Sketch or computer plot this function for $N = 1, 3, 5, 10$ and an arbitrary ω_0. Notice that for $N \gg 1$, $g_{s,2}(\omega)$ may be interpreted as proportional to what we later define as the Dirac delta function, δ, in Boas Chapter 8, Section 11.

(Note: You do not need to provide a mathematically rigorous proof that $g_{s,2}(\omega)$ indeed becomes proportional to $\delta(\omega-\omega_0)$ as $N \to \infty$. For now, observe the shape of $g_{s,2}(\omega)$ as you vary N, and based on your observation simply argue that $\lim_{\omega\to\omega_0} g_{s,2}(\omega) \to \infty$ as $N \to \infty$.)

(e) Show that the first zeros of $g_{s,2}(\omega)$ from $\omega = \omega_0$ are at $\omega = \omega_0 \pm \Delta\omega = \omega_0 \pm \frac{\omega_0}{N}$. Justify that $\Delta\omega = \frac{\omega_0}{N}$ could be a good measure of the spread (or uncertainty) in frequency of our clipped wave train. Then, establish the inverse relationship between the wave train's pulse length ($N\pi$) and the frequency spread ($\Delta\omega$). Finally, using the relationship along with the assumed *wave nature of matter*, explain the uncertainty principle of quantum mechanics.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. Note that the inverse relationship found here is a fundamental property of the finite wave train, and has little to do with any additional *ad hoc* postulates in quantum mechanics. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

10. Let us consider various objects falling under the gravitational acceleration g.

(a) Consider a ball of mass m falling downward. y is the distance the ball traveled at time t (i.e., $y > 0$ and $\hat{\mathbf{e}}_y$ points downward). The ball experiences a resistive force proportional to its speed, $-bv(t)$, where $v(t) = y'(t)$. With the initial condition $v(0) = 0$, show that the speed of the ball is written as

$$v(t) = v_{\rm t}\left(1 - e^{-\frac{b}{m}t}\right),$$

where $v_{\mathrm{t}} = mg/b$ is a terminal velocity as $t \to \infty$.

(b) Now consider a different air drag. A falling parachutist experiences a quadratic resistive force $-bv^2(t)$ on the parachute. For simplicity, assume that the parachute opens immediately at $t = 0$ when $v(0) = 0$. Prove that the speed of the parachutist can be written as

$$v(t) = v_{\mathrm{t}} \tanh\left(\frac{t}{T}\right) ,$$

where $v_{\mathrm{t}} = \sqrt{mg/b}$ is now a different terminal velocity and $T = \sqrt{m/gb}$ is the timescale that characterizes the asymptotic approach of $v(t)$ to v_{t}.

(c) Let us insert numerical values in your answer in (b). For a skydiver in free fall with $m = 70$ kg, find the terminal velocity with the constant of proportionality (friction coefficient) $b = 0.25\,\mathrm{kg\,m^{-1}}$. Then, for the same skydiver but now with her parachute open, find the terminal velocity with $b = 700\,\mathrm{kg\,m^{-1}}$.

(Note: If you are not familiar with the concept of terminal velocity, you may want to review the freshman physics textbooks such as Halliday & Resnick. For (b), you may find it useful to write your equation of motion in a separable form, $\frac{dv}{v_{\mathrm{t}}^2 - v^2} = \frac{bdt}{m}$.)