Mathematical Physics I (Fall 2022): Homework #3

Due Oct. 14, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

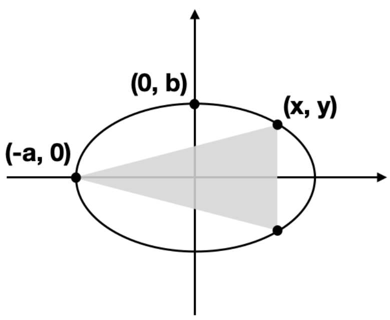
• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.

1.-3. Boas Chapter 4, Problems 7.28, 9.9, 11.7

(Note: For Problem 7.28, you may want to prove and utilize the findings in Problem 7.27. Note the meaning of the subscripts next to the partial derivatives from Boas Chapter 4, Section 1 in case you did not read it. Prove further that the resulting formula is the familiar $c_p - c_v = nR$ found in e.g., Schroeder or Halliday & Resnick, where n is the number of moles of gas present and R is the gas constant. For Problem 9.9, tackle the problem in two ways, first with the substitution method in Section 8 and then with the Lagrange multiplier method in Section 9. For Problem 11.7, notice that the equation we start with is a special case of Eq.(2.1) in Boas Chapter 12, Section 2.)



4.-5. Boas Chapter 5, Problems 4.19, 5.1

(Note: For Problem 4.19, you may want to prove and utilize the first theorem in Problem 4.18. You will also have to show that the intervals of integration for u and v are $[-\infty, \infty]$ and $[0, \infty]$, respectively. See the hint in Problem 4.20 for more information.)

6.-8. Boas Chapter 6, Problems 3.14, 6.16, 9.4

(Note: For Problem 3.14, tackle the problem in two different ways, first by writing in ordinary vector notation, then by writing in tensor notation after reviewing the derivation of Eq.(5.12) in Boas Chapter 10, Section 5.)

9. In the class we discussed the Legendre transformation as one of the examples of a simple change of variables that is found to be useful in classical mechanics and thermodynamics.

(a) Review the general discussion of a Legendre transformation in Boas Chapter 4, Section 11 by following the procedure step by step from Eq.(11.21) to Eq.(11.27).

(b) In one practical example in classical mechanics, given $L(x, \dot{x})$ with $dL = \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial \dot{x}}d\dot{x} = \frac{\partial L}{\partial x}dx + p d\dot{x}$ (note $p \equiv \frac{\partial L}{\partial \dot{x}}$), one can find $H(x, p)$ so that $dH = \frac{\partial L}{\partial x}dx - \dot{x}dp$. Find a Legendre transformation that gives $H(x, p)$. Discuss the meaning of the two functions, L and H, by identifying them as Lagrangian and Hamiltonian, respectively.

(c) In another example in thermodynamics which we briefly examined but left for your exercise, given $U(S, V)$ with $dU = \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial V}dV = TdS - PdV$ (note $T \equiv \left(\frac{\partial S}{\partial U}\right)^{-1}$ and $P \equiv -\frac{\partial U}{\partial V}$), one can find $H(S, P)$ so that $dH = TdS + VdP$. Find a Legendre transformation that gives $H(S, P)$. Discuss the meaning of the two functions, U and H, by identifying them as (internal) energy and enthalpy, respectively.

(d) Given $dU = TdS - pdV$, perform a different Legendre transformation to find another useful function related to energy, $F(T, V)$, known as the Helmholtz free energy. Finally, by performing Legendre transformations on both terms in dU, find yet another function related to energy, $G(T, P)$, known as the Gibbs free energy.

(Note: You may want to briefly review the textbooks in classical mechanics such as Marion, and in statistical mechanics such as Schroeder or Reif. For (b) and (c), you are simply asked to come up with 2-3 sentences about how the these quantities are used in the respective textbooks, without going into all the mechanical or thermodynamical details. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

10. (a) Review Boas Chapter 6, Example 7.2, thus prove Eq.(7.6) or Eq.(f) in p.339. You are also asked to prove the identity by explicitly working with the components, i.e.,

$$\nabla \cdot (\phi \mathbf{V}) = \frac{\partial(\phi V_x)}{\partial x} + \frac{\partial(\phi V_y)}{\partial y} + \frac{\partial(\phi V_z)}{\partial z} = ...$$

[Problem 10 continues in the next page.]

(b) Review how Gauss's law $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ is acquired from Coulomb's law in Boas Chapter 6, Section 10, where $\mathbf{E}$ is the electric field, ρ is the volume charge density, and ϵ_0 is the permittivity of free space.

(c) Prove that the energy of a continuous charge distribution is given by

$$\frac{1}{2}\int \rho V \, d\tau = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2 d\tau$$

for integration over all space, where V is the electrostatic potential. You are first asked to briefly discuss how one came up with the term $\frac{1}{2}\int \rho V \, d\tau$. To prove the equality, you may assume that V vanishes at large distance r at least as fast as r^{-1}.

(Note: The exercises here should sound familiar to most of you as you have begun to study and explore electromagnetism. If not, you may want to briefly review the classic textbooks in electromagnetism such as Griffiths or Jackson.)