Mathematical Physics I (Fall 2022): Homework #2

Due Sep. 30, 2022 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.

1.-8. Boas Chapter 3, Problems 5.42, 6.16, 7.35, 8.16, 9.17, 11.32, 11.60, 12.9 (for 12.4)

(Note: For Problem 5.42, review the Examples in Boas Chapter 3, Section 5 in case you have not done it yet. Find the most relevant example and use the method depicted in there. For Problem 6.16, find the inverse with two different methods — using Eq.(6.13) of Boas Chapter 3 and using the *Gauss-Jordan matrix inversion procedure*. Compare your results with the one found with a computer. In Problem 8.16, there is a typo that is not yet included in the errata list collected by Harold Boas. See if you can find it. For Problem 11.60, you will need to first prove then utilize the findings in Problem 11.57, or Eq.(11.36). For Problem 12.9, consider only Problem 12.4.)

9. In this problem we consider the spin matrices in quantum mechanics that describe particles of various spins in three dimensions.

(a) First, work out Problem 6.6 in Boas Chapter 3. Here, for the Pauli spin matrices introduced to describe particles of spin 1/2,

$$A = \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; B = \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; C = \sigma_z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

you will first need to show that $\sigma_j \sigma_k = \delta_{jk} I_2 + i \sum_l \epsilon_{jkl} \sigma_l$, where δ_{jk} and ϵ_{jkl} are defined in Eq.(9.4) of Boas Chapter 3 and in Eq.(5.3) of Chapter 10, respectively, and I_n is the $n \times n$ unit matrix. Then, it naturally follows that $[\sigma_j, \sigma_k] \equiv \sigma_j \sigma_k - \sigma_k \sigma_j = 2i \sum_l \epsilon_{jkl} \sigma_l$, which is called the *fundamental commutation relation* for angular momentum matrices (or $[\sigma_j, \sigma_k] = 2i\sigma_l$ if $j, k, l = 1, 2, 3$ or a cyclic permutation thereof).

(b) Briefly discuss how these spin matrices are introduced and used in quantum mechanics. Prove that σ_x, σ_y and σ_z are both Hermitian and unitary. Show also that $\sigma^2 \equiv \sum_j \sigma_j^2 = 3I_2$.

(c) Now, using the 3×3 spin matrices that can describe particles of spin 1,

$$M_x = M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_y = M_2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad M_z = M_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

show that $[M_j, M_k] = i \sum_l \epsilon_{jkl} M_l$ and $M^2 \equiv \sum_j M_j^2 = 2I_3$.

(d) Finally, using the 4×4 spin matrices that can describe particles of spin 3/2,

$$M_x = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad M_y = \frac{i}{2} \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}, \quad M_z = \frac{1}{2} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix},$$

show that $[M_j, M_k] = i \sum_l \epsilon_{jkl} M_l$. Briefly explain only in words why we need larger-size matrices to describe particles with a larger spin.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. For (b), you are simply asked to come up with 2-3 sentences about how the matrices are used, without diving into laborious quantum mechanical derivations. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

10. In the class we discussed the Gram-Schmidt orthonormalization process for a linear vector space and for a general vector space.

(a) In one example we briefly examined but left for your exercise (Example 14.6 of Boas Chapter 3), with an inner product defined as

$$\langle f|g\rangle = \int_{-1}^{1} f^*(x)g(x)dx,$$

one can construct a set of orthonormal polynomials P_i that satisfy the orthonormality condition on the interval $-1 \le x \le 1$,

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \delta_{mn}$$

(see also Eq.(8.4) of Chapter 12, but note a different normalization factor). We later identify this set of functions as *Legendre polynomials*. Starting from Eq.(14.10), follow the procedure step by step and find for yourself the first four members of P_i.

[Problem 10 continues in the next page.]

(b) In a similar manner, with an inner product defined as

$$\langle f|g\rangle = \int_0^\infty f^*(x)g(x)e^{-x}dx,$$

find the first three members in the set of orthonormal polynomials L_i that satisfy the orthonormality condition on the interval $0 \le x \le \infty$,

$$\int_0^\infty L_m(x)L_n(x)e^{-x}dx = \delta_{mn}$$

(see also Eq.(22.22) of Chapter 12). We call this set of functions as *Laguerre polynomials.*

(c) Discuss briefly where the (associated) Legendre polynomials and the (associated) Laguerre polynomials appear in quantum mechanics or in other physics research.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. Once again, you are simply asked to come up with 2-3 sentences about how these polynomials are used, not the detailed physical or mathematical discussions. *Astronomy majors* are encouraged to look for the usage of Legendre polynomials in astrophysics research.)