Mathematical Physics I (Fall 2022): Final Exam Solution

Dec. 10, 2022

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

1. (a) [2 pt] The periodic function $f(x)$ below has a period 2:

$$f(x) = \begin{cases} 1 + 2x, & -1 < x < 0 \\ 1 - 2x, & 0 < x < 1. \end{cases}$$

Sketch several periods of $f(x)$. Expand $f(x)$ in a complex exponential Fourier series, and use your result with Dirichlet's theorem and Parseval's theorem to show that

$$\sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$$

(b) [2 pt] Find the Fourier cosine transform of $f(x) = e^{-|x|}$ and show that $f(x)$ is written as

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos \alpha x}{1 + \alpha^2}\, d\alpha\,.$$

Using this result, find the Fourier cosine transform of $h(x) = \frac{1}{1+x^2}$. (Hint: you do not need to perform a messy integration for this.) Show also that

$$\int_0^\infty \frac{d\alpha}{1 + \alpha^2} = \frac{\pi}{2}\,.$$

• (a) From Eq.(8.3) of Boas Chapter 7, you can find

$$
\begin{aligned}
c_0 &= \frac{1}{2}\int_{-1}^{1} f(x)dx = \frac{1}{2}\left\{\int_{-1}^{0}(1+2x)dx + \int_{0}^{1}(1-2x)dx\right\} = 0, \\
c_n &= \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx = \frac{1}{2}\left\{\int_{-1}^{0}(1+2x)e^{-in\pi x}dx + \int_{0}^{1}(1-2x)e^{-in\pi x}dx\right\} \\
&= \frac{1}{2}\left\{\int_{-1}^{1} e^{-in\pi x}dx - 2\int_{0}^{1} x(e^{in\pi x} + e^{-in\pi x})dx\right\} = -2\int_{0}^{1} x\cos n\pi x\, dx \\
&= -2\left\{\left[\frac{x\sin n\pi x}{n\pi}\right]_0^1 + \frac{1}{n\pi}\int_0^1 \sin n\pi x\, dx\right\} = \frac{2[1-(-1)^n]}{n^2\pi^2} = \begin{cases} 0, & \text{if } n \text{ even,} \\ \frac{4}{n^2\pi^2}, & \text{if } n \text{ odd,} \end{cases}
\end{aligned}
$$

which yields

$$
f(x) = \frac{4}{\pi^2}\sum_{\substack{n=-\infty \\ n\,\text{odd}}}^{\infty} \frac{e^{in\pi x}}{n^2} = \frac{4}{\pi^2}\sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{e^{in\pi x} + e^{-in\pi x}}{n^2}.
$$

From Dirichlet's theorem in Boas Chapter 7, Section 6, you reach

$$
1 = f(0) = 2\cdot\frac{4}{\pi^2}\sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty}\frac{1}{n^2}\,,
$$

and from Parseval's theorem in Eq.(11.4) of Boas Chapter 7, you also find

$$
\frac{1}{2}\int_{-1}^{1}|f(x)|^2\,dx = 2\cdot\frac{1}{2}\int_0^1(1-2x)^2\,dx = \frac{1}{3} = \sum_{n=1}^{\infty} c_n^2 = 2\cdot\frac{4^2}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}\,.
$$

• (b) From Eq.(12.15) of Boas Chapter 7, you can find

$$
\begin{aligned}
g_c(\alpha) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\alpha x\,dx = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-x}(e^{i\alpha x} + e^{-i\alpha x})\,dx \\
&= \frac{1}{\sqrt{2\pi}}\left[-\frac{1}{i\alpha - 1} + \frac{1}{i\alpha + 1}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+\alpha^2}\,,
\end{aligned}
$$

which gives

$$
f(x) = e^{-|x|} = \sqrt{\frac{2}{\pi}}\int_0^\infty g_c(\alpha)\cos\alpha x\,d\alpha = \frac{2}{\pi}\int_0^\infty \frac{\cos\alpha x}{1+\alpha^2}\,d\alpha.
$$

It is straightforward to see in this equation that the Fourier cosine transform of $h(x) = \frac{1}{1+x^2}$ is

$$
g_{c,h}(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos\alpha x}{1+x^2}\,dx = \sqrt{\frac{\pi}{2}}e^{-|\alpha|}.
$$

Also, from the Fourier integral theorem in Boas Chapter 7, Section 12, you find

$$
1 = f(0) = \frac{2}{\pi}\int_0^\infty \frac{d\alpha}{1+\alpha^2}\,.
$$

2. Find the general solution to each of the following differential equations.

(a) [1 pt] $x(\ln y)y' - y\ln x = 0$

(b) [2 pt] $y'' - 2y' = 4e^{2x} + 9xe^{-x}$

(c) [1 pt] $xy'' + y' = 4x$

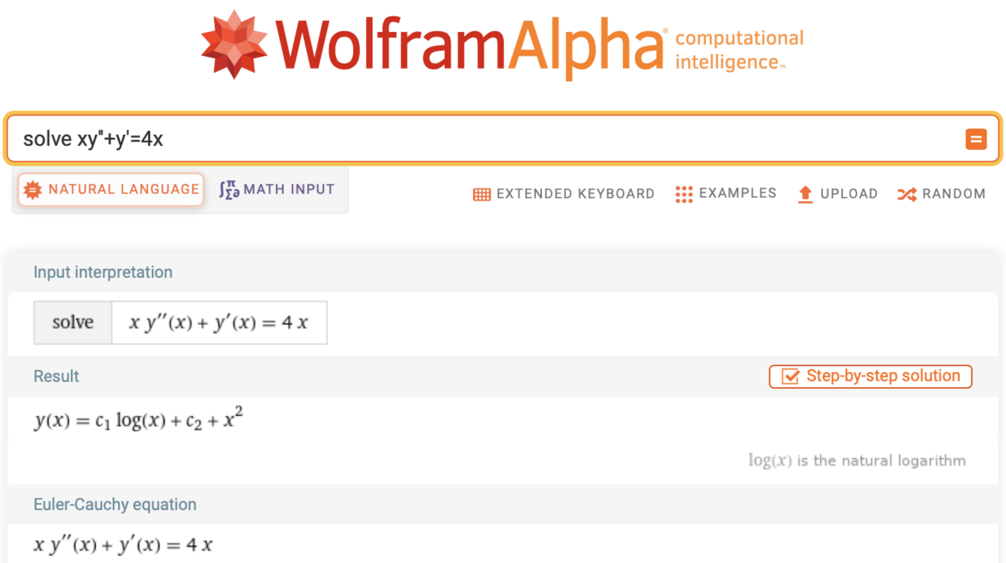
• (a) Rearranging the separable equation given, you get $\frac{\ln y}{y}dy = \frac{\ln x}{x}dx$. Integrating both sides, you reach $(\ln y)^2 = (\ln x)^2 + C$.

• (b-1) The homogeneous equation $y'' - 2y' = 0$ gives the complementary solution $y_c = A + Be^{2x}$. And from Eq.(6.18) of Boas Chapter 8, the inhomogeneous equation $y'' - 2y' = 4e^{2x}$ requires us try a particular solution of the form $y_{p1} = Cxe^{2x}$. Plugging y_{p1} back into the equation yields $C = 2$.

• (b-2) Now to find a particular solution y_{p2} for $y'' - 2y' = 9xe^{-x}$, from Eq.(6.24) of Boas Chapter 8 you try a particular solution of the form $y_{p2} = e^{-x}(Dx + E)$. Plugging this trial solution into the equation, you get $3Dxe^{-x} + (3E - 4D)e^{-x} = 9xe^{-x}$, which leads to $D = 3$ and $E = 4$. Combining all the above, you reach $y = y_c + y_{p1} + y_{p2} = A + Be^{2x} + 2xe^{2x} + (3x + 4)e^{-x}$.

• (c-1) Combining the technique in Eq.(7.2) of Boas Chapter 8, Section 7 (i.e., Case (a)) and in Eq.(3.9) of Boas Chapter 8, Section 3, we attempt to find a general solution of the equation $xp' + p = 4x$ or $p' + \frac{1}{x}p = 4$ with $p = y'$. From Eq.(3.9) of Boas Chapter 8, $I = \int P dx = \int \frac{dx}{x} = \ln x \;\; \rightarrow \;\; p = e^{-I}\int Qe^{I}dx + C_1 e^{-I} = \frac{1}{x}\int 4x dx + \frac{C_1}{x} = 2x + \frac{C_1}{x}$. Therefore, $y = \int p dx = x^2 + C_1 \ln x + C_2$.

• (c-2) Alternatively, you may regard the given equation as the Euler-Cauchy equation discussed in Eqs.(7.17)-(7.19) of Boas Chapter 8, Section 7 (i.e., Case (e)). Inserting $xy' = \frac{dy}{dz}$ and $x^2y'' = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ into the equation $x^2y'' + xy' = 4x^2$ gives $\frac{d^2y}{dz^2} = 4x^2 = 4e^{2z}$. Thus, from $y(z) = y_c + y_p = C_1 z + C_2 + e^{2z}$, you get $y(x) = C_1 \ln x + C_2 + x^2$.



3. (a) [1pt] One of the ordinary differential equations that frequently appears in physics has the form $y'' + f(y) = 0$. Show that the equation is integrated to yield

$$\frac{1}{2}y'^2 + \int f(y)dy = \text{constant}\,.$$

(b) [2 pt] An electron of mass m is accelerated in the electric field of a positively charged sphere. The force between them is inversely proportional to the square of the distance r between the electron and the center of the sphere (with the constant of proportionality k). Find the electron's differential equation of motion. Now, let the electron fall from rest at infinity to the sphere at time $t = 0$. By using the method established in (a), find the electron's velocity $v(r)$ as a function of r before it reaches the surface of the sphere. (Note: Make sure to choose the appropriate sign — e.g., a positive sign indicates outward motion.)

(c) [1 pt] Finally, consider a motion of an electron that is shot radially inward from r_0 at $t = 0$, with initial velocity $v_0 = -\sqrt{\frac{2k}{mr_0}}$. Find the electron's velocity $v(r)$. Utilizing the fact that the equation you found is separable, obtain $r(t)$ or $t(r)$.

- (a) See Eq.(7.13) of Boas Chapter 8, Section 7 (i.e., Case (c)).
- (b) The Newtonian equation of motion is integrated to yield

$$m\frac{d^2r}{dt^2} = -\frac{k}{r^2} \;\rightarrow\; \frac{1}{2}\left(\frac{dr}{dt}\right)^2 = -\int \frac{kdr}{mr^2} + C \;\rightarrow\; \frac{1}{2}[v(r)]^2 - 0 = \left[\frac{k}{mr}\right]_\infty^r = \frac{k}{mr} - \frac{k}{m(\infty)},$$

where the integration constant is found from the initial condition. Therefore, $v(r) = -\sqrt{\frac{2k}{mr}}$.

- (c) Now, integrating the equation of motion inward from r_0 to r,

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = -\int \frac{kdr}{mr^2} + C \;\rightarrow\; \frac{1}{2}[v(r)]^2 - \frac{1}{2}v_0^2 = \left[\frac{k}{mr}\right]_{r_0}^r = \frac{k}{mr} - \frac{k}{mr_0},$$

which leads to $v(r) = -\sqrt{\cancel{v_0^2} + \frac{2k}{m}\left(\frac{1}{r} - \cancel{\frac{1}{r_0}}\right)} = -\sqrt{\frac{2k}{mr}}$ again. Note that the only difference between (b) and (c) is when we set $t = 0$; that is, at $r = \infty$ in (b), but at $r = r_0$ in (c). Thus,

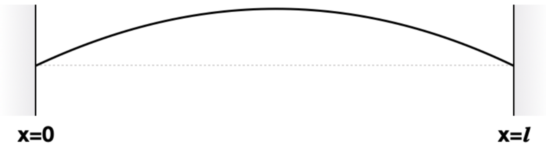
$$v(r) = \frac{dr}{dt} = -\sqrt{\frac{2k}{mr}} \;\rightarrow\; dt = -\sqrt{\frac{mr}{2k}}dr \;\rightarrow\; \int_0^t dt = -\int_{r_0}^r \sqrt{\frac{mr}{2k}}dr$$

$$\rightarrow\; t = t(r) = -\frac{2}{3}\sqrt{\frac{m}{2k}}\left[r^{\frac{3}{2}}\right]_{r_0}^r = \frac{2}{3}\sqrt{\frac{m}{2k}}\left(r_0^{\frac{3}{2}} - r^{\frac{3}{2}}\right).$$

4. (a) [2 pt] The following differential equation describes the *response* of a mechanical system to a unit impulse at time $t = t_0\,(> 0)$. Assuming $y_0 = y_0' = 0$ at $t = 0$ and using the Laplace transform, find the response $y(t)$. (Note: The table of Laplace transforms is in the last page of this exam.)

$$y'' + 2y' + 10y = \delta(t - t_0)$$

(b) [2 pt] A string of length l has a zero initial velocity and the initial displacement $y_0(x) = x(l-x)$ as shown below. Find the displacement as a function of position x, time t, and the wave velocity v that depends on the tension and the linear density of the string.



• (a) With Eqs.(9.1)-(9.2) and (11.7) of Boas Chapter 8, or $L27$ in the Laplace transform table (Boas p.469-471), we transform the given differential equation into

$$p^2Y + 2pY + 10Y = L[\delta(t-t_0)] = e^{-pt_0} \quad \to \quad Y = \frac{e^{-pt_0}}{(p+1)^2 + 3^2}.$$

From $y = L^{-1}(Y)$ and $L13$ and $L28$ in the Laplace transform table, you then acquire

$$y(t) = \begin{cases} \frac{1}{3}e^{-(t-t_0)}\sin 3(t-t_0), & t > t_0 \\ 0, & t < t_0. \end{cases}$$

• (b) We write the solution in the form of Eq.(4.7) of Boas Chapter 13 as

$$y = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}.$$

At $t = 0$ you want

$$y_0(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = x(l-x)$$

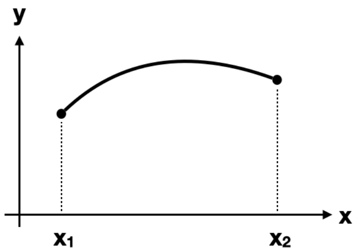
from which you can get the Fourier coefficients as

$$\begin{aligned} b_n &= \frac{2}{l}\int_0^l y_0(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}dx \\ &= \frac{2}{l}\left\{\left[-lx\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l + \cancel{\int_0^l l\frac{l}{n\pi}\cos\frac{n\pi x}{l}dx}\right\} + \frac{2}{l}\left\{\left[x^2\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right]_0^l - \int_0^l 2x\frac{l}{n\pi}\cos\frac{n\pi x}{l}dx\right\} \\ &= \cancel{-\frac{2}{l}\frac{l^3}{n\pi}\cos n\pi x} + \cancel{\frac{2}{l}\frac{l^3}{n\pi}\cos n\pi x} - \frac{4}{n\pi}\left\{\cancel{\left[x\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right]_0^l} - \int_0^l \frac{l}{n\pi}\sin\frac{n\pi x}{l}dx\right\} \\ &= -\frac{4l^2}{n^3\pi^3}(\cos n\pi - 1) = \begin{cases} 0, & \text{if } n \text{ even}, \\ \frac{8l^2}{n^3\pi^3}, & \text{if } n \text{ odd}. \end{cases} \end{aligned}$$

Therefore,

$$y(x,t) = \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{8l^2}{n^3\pi^3}\sin\frac{n\pi x}{l}\cos\frac{n\pi vt}{l}.$$

5. [2 pt] Let us find the light path using Fermat's principle in an atmosphere where the speed of light increases in proportion to the height y (i.e., the index of refraction is proportional to y^{-1}). Write down the transit time from (x_1, y_1) to (x_2, y_2) as an integral, and solve the Euler equation to make the integral stationary. If needed, change the independent variable to make the equation simpler. You will find that this light path is the arc of a circle whose center is on the $y = 0$ line. (Note: This problem simulates a condition for a so-called *superior* mirage — as opposed to an *inferior* mirage — or a condition near a black hole's surface where one can model that the speed of light drastically changes with y, even approaching zero at the *event horizon*.)



Superior mirage (left; 위 신기루) & inferior mirage (right; 아래 신기루) [image credits: nytimes.com, epod.usra.edu]

• As in Example 1 of Boas Chapter 9, Section 3 or in the text at the beginning of Problems in Boas Chapter 9, Section 1, the time of transit t is written as

$$t = c^{-1} \int n ds \;\; \propto \;\; I = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{y} dx.$$

As in Example 3 of Boas Chapter 9, Section 3, with $x' = \frac{dx}{dy} = \frac{1}{y'}$, we change the variable as

$$I = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{y} dx = \int_{y_1}^{y_2} \frac{\sqrt{1+y'^2}}{y} x' dy = \int_{y_1}^{y_2} \frac{\sqrt{x'^2+1}}{y} dy.$$

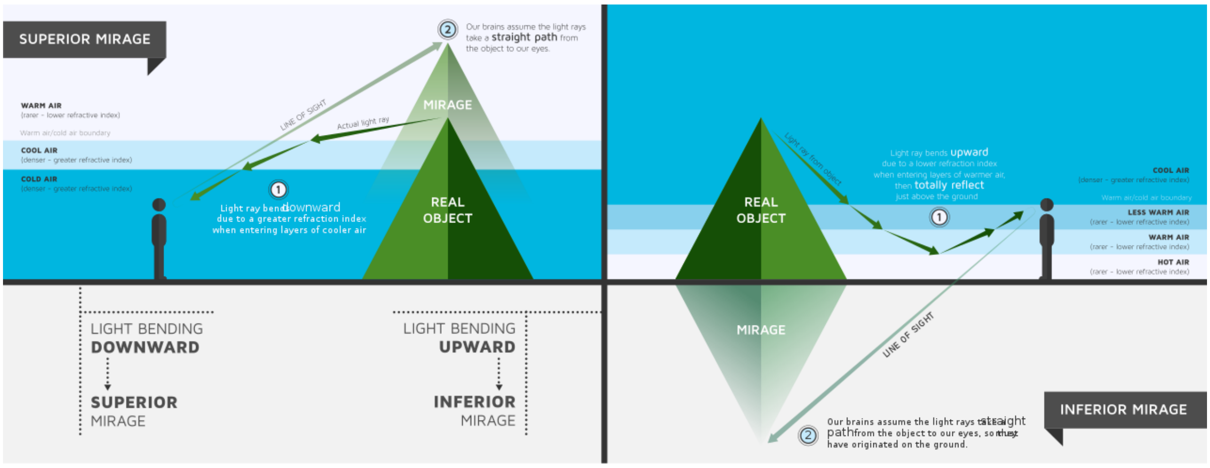
Now, the Euler equation becomes

$$\frac{\partial F}{\partial x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x'} = 0 = \frac{\partial}{\partial y}\left(\frac{x'}{y\sqrt{x'^2+1}}\right) \;\to\; \frac{x'^2}{y^2(x'^2+1)} = C_1^2 \;\to\; x' = \frac{C_1 y}{\sqrt{1-C_1^2 y^2}}.$$

Integrating both sides

$$x = \int \frac{C_1 y dy}{\sqrt{1-C_1^2 y^2}} = -\frac{1}{C_1}\sqrt{1-C_1^2 y^2} + C_2 = -\sqrt{\frac{1}{C_1^2} - y^2} + C_2,$$

from which you acquire $(x - C_2)^2 + y^2 = (\frac{1}{C_1})^2$.



Superior mirage (left) and inferior mirage (right) [image credits: Wikipedia commons]

6. (a) [1 pt] Throughout the semester we discussed many examples in which simple mathematical concepts are utilized to understand seemingly complex physical or daily phenomena. In this regard, five of your peers presented their term projects in the last class of the semester. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 3-4 sentences is expected to clearly convey the core idea of his/her term project. If you were one of the presenters, please choose someone else's.

(b) [1 pt] We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques such as order-of-magnitude estimation and/or dimensional analysis. Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem setup. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in ∼3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.

• (a) See the student presentation slides in Lecture 15-1 that include the collection of term project presentations by five students on December 6.

• (b) See the class slides for Lecture 14-2 that include many example problems, and the grading guideline.