Mathematical Physics I (Fall 2022): Midterm Exam Solution

Oct. 22, 2022

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

1. (a) [1 pt] Find the interval of convergence of the following power series (x is a real number). Be sure to investigate the endpoints of the interval. Note that the series is not in powers of x, but you can transform it into a power series by a change of variable and find where it converges.

$$\sum_{n=0}^{\infty} \frac{3^n(n+1)}{(x+1)^n}$$

(b) [2 pt] The scattering cross section of a photon by an electron was classically derived by J. J. Thomson. Later, Klein and Nishina derived a formula that agrees better with experiments by considering relativistic and quantum mechanical effects in their 1929 paper *"Über die Streuung von Strahlung durch freie Elektronen nach der neuen relativistischen Quantendynamik von Dirac"* (On the scattering of radiation by free electrons according to Dirac's new relativistic quantum dynamics; see their Eq.(61)). The formula contains a term of the form

$$f(\alpha) = \frac{1+\alpha}{\alpha^2}\left[\frac{2(1+\alpha)}{1+2\alpha} - \frac{\ln(1+2\alpha)}{\alpha}\right]$$

where $\alpha = \frac{h\nu}{mc^2}$ is the ratio of the photon energy to the electron rest mass energy. Using a series approximation, find $\lim_{\alpha\to 0} f(\alpha)$.

• (a) $\rho = \lim_{n\to\infty}\left|\frac{3(n+2)}{(x+1)(n+1)}\right| = \frac{3}{|x+1|} < 1$. The endpoints can be studied in the same manner as in Examples of Boas Chapter 1, Section 10, yielding the interval of convergence $x > 2$ or $x < -4$.

• (b) Expanding the terms in binomial power series, $f(\alpha) = \frac{1+\alpha}{\alpha^2}\left[1 + \frac{1}{1+2\alpha} - \frac{\ln(1+2\alpha)}{\alpha}\right] = \frac{1+\alpha}{\alpha^2}\left[1 + \{1 - 2\alpha + (2\alpha)^2 + O(\alpha^3)\} - \frac{1}{\alpha}\{2\alpha - \frac{(2\alpha)^2}{2} + \frac{(2\alpha)^3}{3} + O(\alpha^3)\}\right] = \frac{1+\alpha}{\alpha^2}\cdot\left[\frac{4}{3}\alpha^2 + O(\alpha^3)\right]$, which asymptotes to $\frac{4}{3}$ as $\alpha \to 0$.

2. (a) [1 pt] Find the real part and the imaginary part of $\cosh(2-3i)$. It is expected that your final answer is written in the combinations of trigonometric or hyperbolic functions, or the inverse functions thereof — e.g., $\sin(\square)$, $\sin^{-1}(\square)$, $\sinh(\square)$, $\sinh^{-1}(\square)$ — but nothing else.

(b) [2 pt] Verify that $\tanh^{-1}z = \frac{1}{2}\ln\frac{1+z}{1-z}$ where z is a complex number.

• (a) Using Eqs.(11.4) and (12.2), $\cosh(2-3i) = \frac{1}{2}(e^{2-3i}+e^{-2+3i}) = \frac{1}{2}(e^2\cos 3 + e^{-2}\cos 3) + \frac{i}{2}(-e^2\sin 3 + e^{-2}\sin 3) = \cosh 2\cos 3 - i\sinh 2\sin 3$.

• (b) $\tanh^{-1} z = \omega \;\rightarrow\;$ With $u \equiv e^{\omega}$, you find $z = \tanh\omega = \frac{e^{\omega}-e^{-\omega}}{e^{\omega}+e^{-\omega}} = \frac{u-u^{-1}}{u+u^{-1}} \;\rightarrow\; u^2 = \frac{1+z}{1-z}$ $\rightarrow\; \omega = \ln u = \frac{1}{2}\ln\frac{1+z}{1-z}$.

3. (a) [2 pt] Verify that the following matrix H is Hermitian. Find its eigenvalues and eigenvectors, and write a unitary matrix U which diagonalizes H by a similarity transformation. Finally, show that $U^{-1}HU$ is the diagonal matrix of eigenvalues.

$$H = \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix}$$

(b) [2 pt] Solve the set of equations below by <u>*all three*</u> methods listed here: *(i)* by row reducing the augmented matrix, *(ii)* by using Cramer's rule, *(iii)* by finding the inverse of the coefficient matrix. Clearly indicate which method you are using for each part of your answer, so that the grader could follow it easily.

$$\begin{cases} x - y + z = 4 \\ 2x + y - z = -1 \\ 3x + 2y + 2z = 5 \end{cases}$$

(c) [2 pt] Matrices can be useful wherever linear relations appear — biology, ecology, economics, sociology, statistics, etc. For example, in a study of population movement, the initial fraction of a population in each of n areas can be represented by an n-component column vector $\mathbf{x}$. From its definition, the sum of all n components of $\mathbf{x}$ is 1, i.e., $\sum_{i=1}^{n} x_i = 1$. The movement of people from one area to another in a given time interval is described by an $n \times n$ matrix T. Here T_{ij} is the fraction of the population in the jth area that moves to the ith area. Then, with $\mathbf{x}$ describing the initial population distribution, the final population distribution $\mathbf{x}'$ is given by the matrix equation $\mathbf{x}' = T\mathbf{x}$. Let us assume that the total number of people is unchanged in this time interval. After briefly explaining only in words why we can write

$$\sum_{i=1}^{n} T_{ij} = 1 \qquad (\text{where } j = 1, 2, ..., n),$$

prove explicitly

$$\sum_{i=1}^{n} x_i' = 1.$$

(*Example:* Between the year 2021 and 2022 the population of Korea remained constant with 50 million people. In 2021, Seoul had a population of 10 million. Suppose that the following migration took place between 2021 and 2022: *(i)* 80% of Seoul residents stayed in Seoul, *(ii)* 20% of Seoul residents moved out of Seoul, *(iii)* 90% of residents outside Seoul stayed out, *(iv)* 10% of residents outside Seoul moved to Seoul. Then, the population dynamics in this time period can be described by

$$\begin{pmatrix} 0.24 \\ 0.76 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$$

which is a simple matrix equation, but neatly models the population movement.)

• (a) From the characteristic equation, two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$ appear that correspond to eigenvectors $\mathbf{r}_1 = \frac{1}{\sqrt{3}} \left(\begin{smallmatrix} 1 \\ -1-i \end{smallmatrix} \right)$ and $\mathbf{r}_2 = \frac{1}{\sqrt{3}} \left(\begin{smallmatrix} 1-i \\ 1 \end{smallmatrix} \right)$, respectively. Then it is straightforward to show that

$$U^{-1}HU = \frac{1}{3} \begin{pmatrix} 1 & -1+i \\ 1+i & 1 \end{pmatrix} \begin{pmatrix} 3 & 1-i \\ 1+i & 2 \end{pmatrix} \begin{pmatrix} 1 & 1-i \\ -1-i & 1 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

• (b-1) As in Example 1 of Boas Chapter 3, Section 2,

$$\begin{pmatrix} 1 & -1 & 1 & 4 \\ 2 & 1 & -1 & -1 \\ 3 & 2 & 2 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 3 & -3 & -9 \\ 0 & 5 & -1 & -7 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 3 & -3 & -9 \\ 0 & 0 & 4 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 1 & 4 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

• (b-2) As in Example 5 of Boas Chapter 3, Section 3,

$$x = \begin{vmatrix} 4 & -1 & 1 \\ -1 & 1 & -1 \\ 5 & 2 & 2 \end{vmatrix} \Big/ \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 1 \\ 3 & 0 & 0 \\ -3 & 4 & 0 \end{vmatrix} \Big/ \begin{vmatrix} 1 & -1 & 1 \\ 3 & 0 & 0 \\ 1 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ -3 & 4 \end{vmatrix} \Big/ \begin{vmatrix} 3 & 0 \\ 1 & 4 \end{vmatrix} = 1, \text{ etc.}$$

• (b-3) As in Example 3 of Boas Chapter 3, Section 6,

$$M = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & 2 \end{pmatrix} \to \det(M) = 12 \text{ and } C = \begin{pmatrix} 4 & -7 & 1 \\ 4 & -1 & -5 \\ 0 & 3 & 3 \end{pmatrix} \to M^{-1} = \frac{1}{12} \begin{pmatrix} 4 & 4 & 0 \\ -7 & -1 & 3 \\ 1 & -5 & 3 \end{pmatrix}$$

• (c) By replacing x'_i with $\sum_{j=1}^{n} T_{ij} x_j$,

$$\sum_{i=1}^{n} x'_i = \sum_{i=1}^{n} \sum_{j=1}^{n} T_{ij} x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} T_{ij} \right) x_j = \sum_{j=1}^{n} x_j = 1.$$

4. (a) [1 pt] A scalar function at a point (x, y, z) is given by $T(x, y, z) = y^2 + xz$. Using the Lagrange multiplier method, find the largest and smallest values of T on the surface of a sphere $x^2 + y^2 + z^2 = 1$.

(b) [2 pt] In the kinetic theory of gases we often need to evaluate the integrals of the following forms. By successively differentiating the integral, show that

$$\int_0^\infty t^{2n+1} e^{-\alpha t^2} dt = \frac{n!}{2\alpha^{n+1}}$$

for an integer $n \geq 0$. Here you may want to first explicitly evaluate the case for $n = 0$. Then, show also that

$$\int_0^\infty t^{2n} e^{-\alpha t^2} dt = \frac{\sqrt{\pi}}{2} \frac{(2n-1)!!}{2^n \alpha^{n+\frac{1}{2}}}$$

for an integer $n \geq 1$, given that $\int_0^\infty e^{-\alpha t^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$. Here, $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

• (a) From Eq.(9.6) of Boas Chapter 4, we write $F = y^2 + xz + \lambda(x^2 + y^2 + z^2)$. Then, $\frac{\partial F}{\partial x} = z + 2\lambda x = 0$, $\frac{\partial F}{\partial z} = x + 2\lambda z = 0$, and $\frac{\partial F}{\partial y} = 2y + 2\lambda y = 0$. The last equation offers two possibilities of extremum conditions: (i) $\lambda = -1 \;\rightarrow\; x = z = 0 \;\rightarrow\; y = \pm 1$ (giving $T_{\text{max, local}} = 1$). (ii) $y = 0 \;\rightarrow\; x = \pm\frac{1}{\sqrt{2}} = z$ with $\lambda = -\frac{1}{2}$ (giving $T_{\text{max, local}} = \frac{1}{2}$), or $x = \pm\frac{1}{\sqrt{2}} = -z$ with $\lambda = \frac{1}{2}$ (giving $T_{\text{min, local}} = -\frac{1}{2}$). Comparing all the results, $T_{\text{max}} = 1$ and $T_{\text{min}} = -\frac{1}{2}$.

• (b-1) The first part of this problem is discussed in Example 4 of Boas Chapter 4, Section 12.

• (b-2) By differentiating $\int_0^\infty e^{-\alpha t^2} dt = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$ with respect to α,

$$\frac{dI}{d\alpha} = -\int_0^\infty t^2 e^{-\alpha t^2} dt = \frac{\sqrt{\pi}}{2}\left(-\frac{1}{2}\right)\frac{1}{\alpha^{\frac{3}{2}}}.$$

Repeating the differentiation, one can acquire the integral of any even power of t times $e^{-\alpha t^2}$.

5. (a) [1 pt] Evaluate the integral

$$I = \int_0^1 dy \int_{y^2}^1 \frac{e^x}{\sqrt{x}} dx\,.$$

(b) [1 pt] Evaluate the integral

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} e^{-x^2-y^2} dy\,.$$

• (a) The equivalent integral with the integration in the opposite order is

$$I = \int_0^1 \frac{e^x}{\sqrt{x}} dx \int_0^{\sqrt{x}} dy = \int_0^1 e^x dx = e - 1.$$

• (b) With $x = r\cos\theta$, $y = r\sin\theta$, and $J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$,

$$I = \int_0^{\frac{\pi}{2}} \int_0^1 e^{-r^2} r dr d\theta = \frac{\pi}{2}\int_0^1 e^{-r^2} r dr = \frac{\pi}{4}\left(1 - \frac{1}{e}\right).$$

6. (a) [2 pt] A force field is described by

$$\mathbf{F} = -\frac{y\,\hat{\mathbf{e}}_x}{x^2+y^2} + \frac{x\,\hat{\mathbf{e}}_y}{x^2+y^2}\,.$$

Express $\mathbf{F}$ in cylindrical coordinates (r, θ, z). Then, operating in cylindrical coordinates calculate the work done by $\mathbf{F}$ moving on a unit circle in the xy-plane for the following three cases: *(i)* moving counterclockwise from $\theta = 0$ to π, *(ii)* moving clockwise from $\theta = 0$ to $-\pi$, *(iii)* encircling the unit circle once counterclockwise. What do these observations tells us about the force $\mathbf{F}$ being conservative or nonconservative? If you believe $\mathbf{F}$ is described by $\mathbf{F} = -\nabla\psi$, find $\psi(r, \theta, z)$; otherwise, simply state that no such ψ exists. Finally, examine $\nabla\times\mathbf{F}$. Reconcile any contradiction, if you find any. (Note: You may use the following formula to acquire the gradient or the curl in curvilinear coordinates:

$$\nabla u = \sum_{i=1}^{3} \hat{\mathbf{e}}_i \frac{1}{h_i}\frac{\partial u}{\partial x_i} \quad \text{and} \quad \nabla\times\mathbf{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1V_1 & h_2V_2 & h_3V_3 \end{vmatrix},$$

when the element of arc length is given by $d\mathbf{s} = \sum_{i=1}^{3} \hat{\mathbf{e}}_i h_i dx_i$.)

(b) [1 pt] Prove that $\nabla\cdot(\mathbf{U}\times\mathbf{V}) = \mathbf{V}\cdot(\nabla\times\mathbf{U}) - \mathbf{U}\cdot(\nabla\times\mathbf{V})$. Here you are asked to prove the identity by explicitly working with the components, i.e.,

$$\nabla\cdot(\mathbf{U}\times\mathbf{V}) = \frac{\partial(U_yV_z - U_zV_y)}{\partial x} + \frac{\partial(U_zV_x - U_xV_z)}{\partial y} + \frac{\partial(U_xV_y - U_yV_x)}{\partial z} = \ldots$$

You are also welcomed to tackle this problem using tensor notation for an additional +1 point.

• (a-1) Using Eq.(8.8) of Boas Chapter 10, it is straightforward to show $\mathbf{F} = \hat{\mathbf{e}}_\theta/r$. Then for the unit circle in the xy-plane (i.e., $dr = dz = 0$), you can get $\oint \mathbf{F}\cdot d\mathbf{s} = \int_0^{2\pi} F_\theta r d\theta = \int_0^{2\pi} d\theta = 2\pi$. This tells us that $\mathbf{F}$ is not conservative, according to Thm.(11.10) of Boas Chapter 6. Thus, no acceptable ψ exists that gives $\mathbf{F} = -\nabla\psi$.

• (a-2) From the $\nabla\times\mathbf{V}$ formula given in the problem (or from Example 4 of Boas Chapter 10, Section 9), you get

$$\nabla\times\mathbf{F} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & r\cdot\frac{1}{r} & 0 \end{vmatrix} = \frac{0}{r},$$

which is 0 everywhere except at the origin, $r = 0$ (not defined). Because we cannot say $\nabla\times\mathbf{F} = 0$ at every point of the region, $\mathbf{F}$ is not conservative, again from Thm.(11.10) of Boas Chapter 6.

• (b-1) Working with the vector components,

$$\begin{aligned}
\nabla\cdot(\mathbf{U}\times\mathbf{V}) &= \frac{\partial(U_yV_z - U_zV_y)}{\partial x} + \frac{\partial(U_zV_x - U_xV_z)}{\partial y} + \frac{\partial(U_xV_y - U_yV_x)}{\partial z} \\
&= -U_x\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - U_y\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) - U_z\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \\
&\quad + V_x\left(\frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z}\right) + V_y\left(\frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x}\right) + V_z\left(\frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}\right) \\
&= \mathbf{V}\cdot(\nabla\times\mathbf{U}) - \mathbf{U}\cdot(\nabla\times\mathbf{V}).
\end{aligned}$$

• (b-2) Alternatively, writing in tensor notation and then using Eqs. (5.11), (5.13) and others of Boas Chapter 10,

$$\begin{aligned}
\nabla \cdot (\mathbf{U} \times \mathbf{V}) &= \frac{\partial}{\partial x_i} (\epsilon_{ijk} U_j V_k) = \epsilon_{ijk} \left(\frac{\partial U_j}{\partial x_i} V_k \right) + \epsilon_{ijk} \left(\frac{\partial V_k}{\partial x_i} U_j \right) \\
&= V_k \cdot \epsilon_{kij} \left(\frac{\partial U_j}{\partial x_i} \right) - U_j \cdot \epsilon_{jik} \left(\frac{\partial V_k}{\partial x_i} \right) \\
&= V_k (\nabla \times \mathbf{U})_k - U_j (\nabla \times \mathbf{V})_j \\
&= \mathbf{V} \cdot (\nabla \times \mathbf{U}) - \mathbf{U} \cdot (\nabla \times \mathbf{V}).
\end{aligned}$$