Mathematical Physics I (Fall 2022): Final Examination

Dec. 10, 2022

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

• First, make sure you have all 6 answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from 1 to 6 on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet 2). After the exam, you will separately turn in all 6 answer sheets, even if some sheets are still blank.

• Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you find any issue or question, you *must* raise it in the first 30 minutes. You have to stay in the room for that 30 minutes even if you have nothing to write down.

• Make your writing easy to read. Illegible answers will *not* be graded.

1. (a) [2 pt] The periodic function $f(x)$ below has a period 2:

$$f(x) = \begin{cases} 1 + 2x, & -1 < x < 0 \\ 1 - 2x, & 0 < x < 1. \end{cases}$$

Sketch several periods of $f(x)$. Expand $f(x)$ in a complex exponential Fourier series, and use your result with Dirichlet's theorem and Parseval's theorem to show that

$$\sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{\substack{n=1 \\ n\,\text{odd}}}^{\infty} \frac{1}{n^4} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$$

(b) [2 pt] Find the Fourier cosine transform of $f(x) = e^{-|x|}$ and show that $f(x)$ is written as

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\cos \alpha x}{1 + \alpha^2}\, d\alpha\,.$$

Using this result, find the Fourier cosine transform of $h(x) = \frac{1}{1+x^2}$. (Hint: you do not need to perform a messy integration for this.) Show also that

$$\int_0^\infty \frac{d\alpha}{1 + \alpha^2} = \frac{\pi}{2}\,.$$

2. Find the general solution to each of the following differential equations.

(a) [1 pt] $x(\ln y)y' - y \ln x = 0$

(b) [2 pt] $y'' - 2y' = 4e^{2x} + 9xe^{-x}$

(c) [1 pt] $xy'' + y' = 4x$

3. (a) [1pt] One of the ordinary differential equations that frequently appears in physics has the form $y'' + f(y) = 0$. Show that the equation is integrated to yield

$$\frac{1}{2}y'^2 + \int f(y)dy = \text{constant}\,.$$

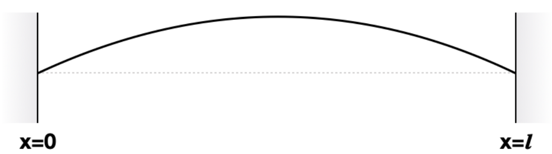
(b) [2 pt] An electron of mass m is accelerated in the electric field of a positively charged sphere. The force between them is inversely proportional to the square of the distance r between the electron and the center of the sphere (with the constant of proportionality k). Find the electron's differential equation of motion. Now, let the electron fall from rest at infinity to the sphere at time $t = 0$. By using the method established in (a), find the electron's velocity $v(r)$ as a function of r before it reaches the surface of the sphere. (Note: Make sure to choose the appropriate sign — e.g., a positive sign indicates outward motion.)

(c) [1 pt] Finally, consider a motion of an electron that is shot radially inward from r_0 at $t = 0$, with initial velocity $v_0 = -\sqrt{\frac{2k}{mr_0}}$. Find the electron's velocity $v(r)$. Utilizing the fact that the equation you found is separable, obtain $r(t)$ or $t(r)$.

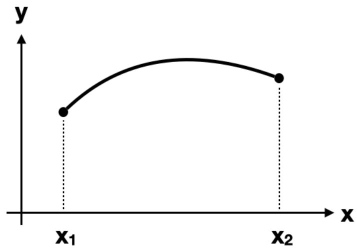
4. (a) [2 pt] The following differential equation describes the *response* of a mechanical system to a unit impulse at time $t = t_0\,(> 0)$. Assuming $y_0 = y_0' = 0$ at $t = 0$ and using the Laplace transform, find the response $y(t)$. (Note: The table of Laplace transforms is in the last page of this exam.)

$$y'' + 2y' + 10y = \delta(t - t_0)$$

(b) [2 pt] A string of length l has a zero initial velocity and the initial displacement $y_0(x) = x(l - x)$ as shown below. Find the displacement as a function of position x, time t, and the wave velocity v that depends on the tension and the linear density of the string.



5. [2 pt] Let us find the light path using Fermat's principle in an atmosphere where the speed of light increases in proportion to the height y (i.e., the index of refraction is proportional to y^{-1}). Write down the transit time from (x_1, y_1) to (x_2, y_2) as an integral, and solve the Euler equation to make the integral stationary. If needed, change the independent variable to make the equation simpler. You will find that this light path is the arc of a circle whose center is on the $y = 0$ line. (Note: This problem simulates a condition for a so-called *superior* mirage — as opposed to an *inferior* mirage — or a condition near a black hole's surface where one can model that the speed of light drastically changes with y, even approaching zero at the *event horizon*.)



Superior mirage (left; 위 신기루) & inferior mirage (right; 아래 신기루) [image credits: nytimes.com, epod.usra.edu]

6. (a) [1 pt] Throughout the semester we discussed many examples in which simple mathematical concepts are utilized to understand seemingly complex physical or daily phenomena. In this regard, five of your peers presented their term projects in the last class of the semester. Describe the key idea of one of the presentations you found interesting. A paragraph of at least 3-4 sentences is expected to clearly convey the core idea of his/her term project. If you were one of the presenters, please choose someone else's.

(b) [1 pt] We also discussed how one can speedily gain insights into a physical phenomenon, by using techniques such as order-of-magnitude estimation and/or dimensional analysis. Invent and solve your own order-of-magnitude estimation problem. Start with a paragraph of at least 2-3 sentences to clearly describe the problem setup. Make a physically intuitive, yet simple problem so that you can explain your problem *and* solution to a fellow physics major student in ∼3 minutes. Use diagrams if desired. Do not plagiarize another person's idea.