

# Mathematical Physics I (Fall 2022): Midterm Examination

Oct. 22, 2022

[total 20 pts, closed book/cellphone, no calculator, 90 minutes]

- First, make sure you have all 6 answer sheets. Write down your name and student ID on each of all 6 answer sheets. Then, number the sheets from (1) to (6) on the top right corner. Your answer to each problem must *only* be in the sheet with the matching number (e.g., your answer to Problem 2 must *only* be in sheet (2)). After the exam, you will separately turn in all 6 answer sheets, even if some sheets are still blank.
- Make sure you have all 6 problems. Have a quick look through them all and portion your time wisely. If you find any issue or question, you *must* raise it in the first 30 minutes. You have to stay in the room for that 30 minutes even if you have nothing to write down.
- Make your writing easy to read. Illegible answers will *not* be graded.

1. (a) [1 pt] Find the interval of convergence of the following power series ( $x$  is a real number). Be sure to investigate the endpoints of the interval. Note that the series is not in powers of  $x$ , but you can transform it into a power series by a change of variable and find where it converges.

$$\sum_{n=0}^{\infty} \frac{3^n(n+1)}{(x+1)^n}$$

(b) [2 pt] The scattering cross section of a photon by an electron was classically derived by J. J. Thomson. Later, Klein and Nishina derived a formula that agrees better with experiments by considering relativistic and quantum mechanical effects in their 1929 paper “*Über die Streuung von Strahlung durch freie Elektronen nach der neuen relativistischen Quantendynamik von Dirac*” (On the scattering of radiation by free electrons according to Dirac’s new relativistic quantum dynamics; see their Eq.(61)). The formula contains a term of the form

$$f(\alpha) = \frac{1+\alpha}{\alpha^2} \left[ \frac{2(1+\alpha)}{1+2\alpha} - \frac{\ln(1+2\alpha)}{\alpha} \right]$$

where  $\alpha = \frac{h\nu}{mc^2}$  is the ratio of the photon energy to the electron rest mass energy. Using a series approximation, find  $\lim_{\alpha \rightarrow 0} f(\alpha)$ .

2. (a) [1 pt] Find the real part and the imaginary part of  $\cosh(2 - 3i)$ . It is expected that your final answer is written in the combinations of trigonometric or hyperbolic functions, or the inverse functions thereof — e.g.,  $\sin(\square)$ ,  $\sin^{-1}(\square)$ ,  $\sinh(\square)$ ,  $\sinh^{-1}(\square)$  — but nothing else.

(b) [2 pt] Verify that  $\tanh^{-1}z = \frac{1}{2} \ln \frac{1+z}{1-z}$  where  $z$  is a complex number.

3. (a) [2 pt] Verify that the following matrix  $H$  is Hermitian. Find its eigenvalues and eigenvectors, and write a unitary matrix  $U$  which diagonalizes  $H$  by a similarity transformation. Finally, show that  $U^{-1}HU$  is the diagonal matrix of eigenvalues.

$$H = \begin{pmatrix} 3 & 1 - i \\ 1 + i & 2 \end{pmatrix}$$

(b) [2 pt] Solve the set of equations below by all three methods listed here: (i) by row reducing the augmented matrix, (ii) by using Cramer's rule, (iii) by finding the inverse of the coefficient matrix. Clearly indicate which method you are using for each part of your answer, so that the grader could follow it easily.

$$\begin{cases} x - y + z = 4 \\ 2x + y - z = -1 \\ 3x + 2y + 2z = 5 \end{cases}$$

(c) [2 pt] Matrices can be useful wherever linear relations appear — biology, ecology, economics, sociology, statistics, etc. For example, in a study of population movement, the initial fraction of a population in each of  $n$  areas can be represented by an  $n$ -component column vector  $\mathbf{x}$ . From its definition, the sum of all  $n$  components of  $\mathbf{x}$  is 1, i.e.,  $\sum_{i=1}^n x_i = 1$ . The movement of people from one area to another in a given time interval is described by an  $n \times n$  matrix  $T$ . Here  $T_{ij}$  is the fraction of the population in the  $j$ th area that moves to the  $i$ th area. Then, with  $\mathbf{x}$  describing the initial population distribution, the final population distribution  $\mathbf{x}'$  is given by the matrix equation  $\mathbf{x}' = T\mathbf{x}$ . Let us assume that the total number of people is unchanged in this time interval. After briefly explaining only in words why we can write

$$\sum_{i=1}^n T_{ij} = 1 \quad (\text{where } j = 1, 2, \dots, n),$$

prove explicitly

$$\sum_{i=1}^n x'_i = 1.$$

(*Example:* Between the year 2021 and 2022 the population of Korea remained constant with 50 million people. In 2021, Seoul had a population of 10 million. Suppose that the following

migration took place between 2021 and 2022: (i) 80% of Seoul residents stayed in Seoul, (ii) 20% of Seoul residents moved out of Seoul, (iii) 90% of residents outside Seoul stayed out, (iv) 10% of residents outside Seoul moved to Seoul. Then, the population dynamics in this time period can be described by

$$\begin{pmatrix} 0.24 \\ 0.76 \end{pmatrix} = \begin{pmatrix} 0.8 & 0.1 \\ 0.2 & 0.9 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.8 \end{pmatrix}$$

which is a simple matrix equation, but neatly models the population movement.)

4. (a) [1 pt] A scalar function at a point  $(x, y, z)$  is given by  $T(x, y, z) = y^2 + xz$ . Using the Lagrange multiplier method, find the largest and smallest values of  $T$  on the surface of a sphere  $x^2 + y^2 + z^2 = 1$ .

(b) [2 pt] In the kinetic theory of gases we often need to evaluate the integrals of the following forms. By successively differentiating the integral, show that

$$\int_0^\infty t^{2n+1} e^{-\alpha t^2} dt = \frac{n!}{2\alpha^{n+1}}$$

for an integer  $n \geq 0$ . Here you may want to first explicitly evaluate the case for  $n = 0$ . Then, show also that

$$\int_0^\infty t^{2n} e^{-\alpha t^2} dt = \frac{\sqrt{\pi} (2n-1)!!}{2 \cdot 2^n \alpha^{n+\frac{1}{2}}}$$

for an integer  $n \geq 1$ , given that  $\int_0^\infty e^{-\alpha t^2} dt = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$ . Here,  $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$ .

5. (a) [1 pt] Evaluate the integral

$$I = \int_0^1 dy \int_{y^2}^1 \frac{e^x}{\sqrt{x}} dx.$$

(b) [1 pt] Evaluate the integral

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} e^{-x^2-y^2} dy.$$

[Problem 6 in the next page.]

6. (a) [2 pt] A force field is described by

$$\mathbf{F} = -\frac{y \hat{\mathbf{e}}_x}{x^2 + y^2} + \frac{x \hat{\mathbf{e}}_y}{x^2 + y^2}.$$

Express  $\mathbf{F}$  in cylindrical coordinates  $(r, \theta, z)$ . Then, operating in cylindrical coordinates calculate the work done by  $\mathbf{F}$  moving on a unit circle in the  $xy$ -plane for the following three cases: (i) moving counterclockwise from  $\theta = 0$  to  $\pi$ , (ii) moving clockwise from  $\theta = 0$  to  $-\pi$ , (iii) encircling the unit circle once counterclockwise. What do these observations tell us about the force  $\mathbf{F}$  being conservative or nonconservative? If you believe  $\mathbf{F}$  is described by  $\mathbf{F} = -\nabla\psi$ , find  $\psi(r, \theta, z)$ ; otherwise, simply state that no such  $\psi$  exists. Finally, examine  $\nabla \times \mathbf{F}$ . Reconcile any contradiction, if you find any. (Note: You may use the following formula to acquire the gradient or the curl in curvilinear coordinates:

$$\nabla u = \sum_{i=1}^3 \hat{\mathbf{e}}_i \frac{1}{h_i} \frac{\partial u}{\partial x_i} \quad \text{and} \quad \nabla \times \mathbf{V} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix},$$

when the element of arc length is given by  $ds = \sum_{i=1}^3 \hat{\mathbf{e}}_i h_i dx_i$ .)

(b) [1 pt] Prove that  $\nabla \cdot (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot (\nabla \times \mathbf{U}) - \mathbf{U} \cdot (\nabla \times \mathbf{V})$ . Here you are asked to prove the identity by explicitly working with the components, i.e.,

$$\nabla \cdot (\mathbf{U} \times \mathbf{V}) = \frac{\partial(U_y V_z - U_z V_y)}{\partial x} + \frac{\partial(U_z V_x - U_x V_z)}{\partial y} + \frac{\partial(U_x V_y - U_y V_x)}{\partial z} = \dots$$

You are also welcomed to tackle this problem using tensor notation for an additional +1 point.