Mathematical Physics I (Fall 2021): Homework #5 Solution

Due Nov. 19, 2021 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

1. Boas Chapter 8, Problem 5.9

(Note: For Problems in Sections 5 and 6, read the instruction in the textbook carefully; that is, you have been asked to find a computer solution and reconcile differences, if any.)

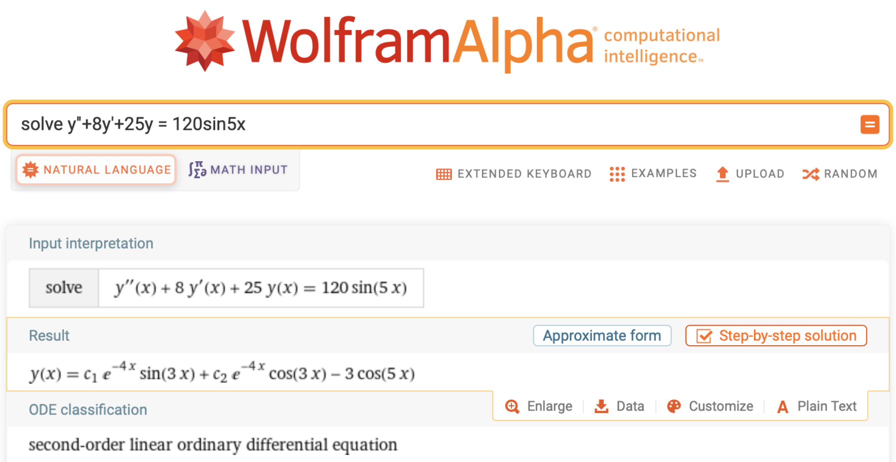
• From Eqs.(5.16)-(5.17), and with $2 \pm 3i$ as the roots of the auxiliary equation, you get $y = e^{2x}(Ae^{i3x} + Be^{-i3x})$, or $e^{2x}(C\sin 3x + D\cos 3x)$, or $y_0 e^{2x}\sin(3x + \gamma)$.

2. Boas Chapter 8, Problem 6.8

• From Eq.(6.18), and with ± 4 as the roots of the auxiliary equation of its homogeneous counterpart, you get $y = y_c + y_p = Ae^{4x} + Be^{-4x} + Cxe^{4x}$. Plugging y_p back into the equation yields $C = 5$.

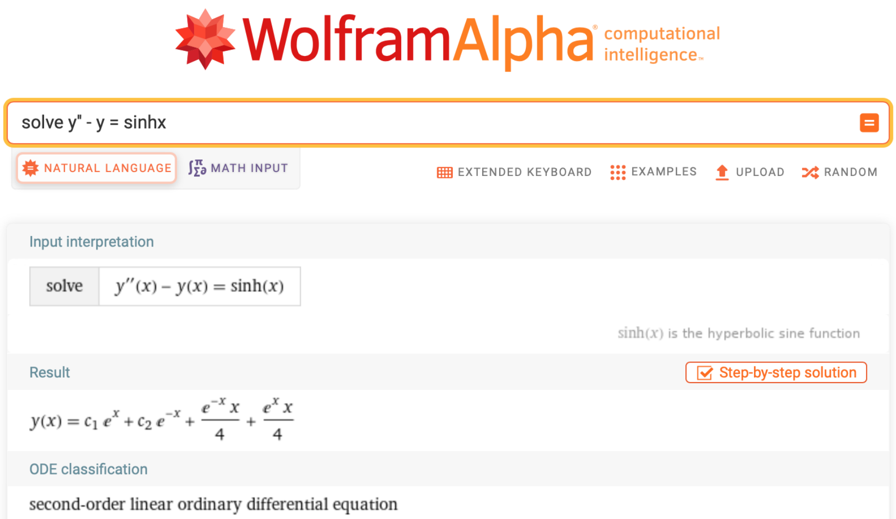
3. Boas Chapter 8, Problem 6.14

• As in Example 6 of Section 6, $Y'' + 8Y' + 25Y = 120e^{i5x}$ gives $Y_p = -3ie^{i5x}$. From $y_p = Im(Y_p)$, you reach $y = y_c + y_p = e^{-4x}(A\sin 3x + B\cos 3x) - 3\cos 5x$.



4. Boas Chapter 8, Problem 6.35

• As in Example 9 of Section 6, $(D^2 - 1)y = \frac{1}{2}(e^x - e^{-x})$ gives $y = y_c + y_{p1} + y_{p2} = Ae^x + Be^{-x} + \frac{1}{4}(xe^x + xe^{-x}) = Ae^x + Be^{-x} + \frac{1}{2}x\cosh x$.



5. Boas Chapter 8, Problem 7.3

(Note: You may continue to utilize computer solutions to validate your answers to problems in Sections 7 to 9.)

• From Eq.(7.3) for Case (b), inserting $y' = p$ and $y'' = p\frac{dp}{dy}$ into the given differential equation gives $2y \cdot p\frac{dp}{dy} = p^2 \rightarrow \frac{2dp}{p} = \frac{dy}{y} \rightarrow \ln p^2 = \ln y + C_1 \rightarrow p^2 = C_2|y| = \left(\frac{dy}{dx}\right)^2 \rightarrow \frac{dy}{|y|^{1/2}} = C_3 dx \rightarrow 2|y|^{\frac{1}{2}} = C_3 x + C_4 \rightarrow y = A(x+B)^2$. This seemingly "general" solution does not include an obvious solution by inspection, $y =$ constant, as we discussed in Example 3 of Section 2.

6. Boas Chapter 8, Problem 7.17

• From Eqs.(7.18)-(7.19) for Case (d), inserting $xy' = \frac{dy}{dz}$ and $x^2y'' = \frac{d^2y}{dz^2} - \frac{dy}{dz}$ into the given differential equation gives $\frac{d^2y}{dz^2} - 16y = 8x^4 = 8e^{4z}$ — which became very similar to Problem 6.8 above. Thus, from $y(z) = y_c + y_p = Ae^{4z} + Be^{-4z} + ze^{4z}$, you get $y(x) = Ax^4 + Bx^{-4} + x^4 \ln x$.

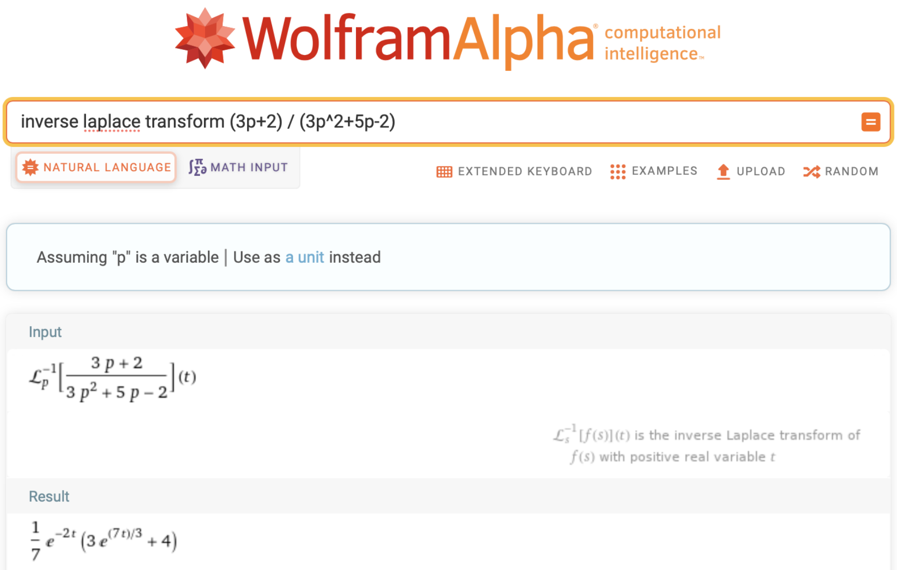
7. Boas Chapter 8, Problem 8.11

• In order to utilize $L7$-$L8$ in the Laplace transform table (Boas p.469-471), we reshape the given formula as

$$Y = \frac{3p+2}{3p^2+5p-2} = \frac{3p+2}{(3p-1)(p+2)} = \frac{p+\frac{2}{3}}{(p-\frac{1}{3})(p+2)} = \frac{p}{(p-\frac{1}{3})(p+2)} + \frac{\frac{2}{3}}{(p-\frac{1}{3})(p+2)}.$$

Therefore, $y = L^{-1}(Y)$ is found to be

$$y = \frac{-\frac{1}{3}e^{t/3} - 2e^{-2t}}{-\frac{1}{3}-2} + \frac{\frac{2}{3}(e^{t/3}-e^{-2t})}{2+\frac{1}{3}} = \frac{3}{7}\left(\frac{1}{3}e^{t/3} + 2e^{-2t} + \frac{2}{3}e^{t/3} - \frac{2}{3}e^{-2t}\right) = \frac{3}{7}e^{t/3} + \frac{4}{7}e^{-2t}.$$



8. Boas Chapter 8, Problem 9.14

• With Eqs.(9.1)-(9.2) and $L6$ in the Laplace transform table, we transform the given differential equation into

$$p^2Y - py_0 - y_0' - 4pY + 4y_0 = L(-4te^{2t}) \;\rightarrow\; p^2Y - 4pY - 1 = -\frac{4}{(p-2)^2} \;\rightarrow\; Y = \frac{1}{(p-2)^2},$$

from which you acquire $y = L^{-1}(Y) = te^{2t}$.

9. Let us apply the Laplace transform to a few physical examples.

(a) Consider a mass m attached to one end of an ideal, massless spring of spring constant k. The free end of the spring is fixed in space and the mass is oscillating under the spring's influence. The displacement $x(t)$ from the equilibrium point (see figure) satisfies the equation of motion

$$m\ddot{x}(t) + kx(t) = 0,$$

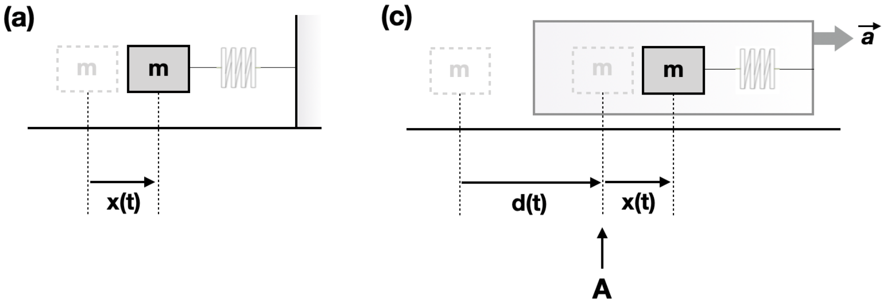
with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Using the Laplace transforms, find $x(t)$.

(b) We then subject the system in (a) to the damping proportional to the velocity as

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0,$$

with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Using the Laplace transforms, recover $x(t)$ that you found in classical mechanics classes, e.g., Eq.(3.40) of Thornton & Marion (5th ed.). Assume underdamped oscillation, i.e., $b^2 < 4mk$.

(c) We now go back to the setup in (a) again. Starting at $t = 0$, the free end of the unstretched spring experiences a constant acceleration **a**, away from the mass m at rest. Using the Laplace transforms, find the position of m. Also find its form in the limit of small t.



(Note: The differential equations here should look familiar to most of you from your classical mechanics classes. If not, you may want to briefly review the textbooks such as Thornton & Marion or Taylor. For (c), the motion of m can be divided into two parts: the accelerated motion of point A defined as where m would have been had the spring been replaced by a string (see figure) + the oscillation of m about A — expressed as $d(t)$ and $x(t)$, respectively.)

• (a) Using Eqs. (9.1)-(9.2) of Boas Chapter 8, you get $p^2X - px_0 + \omega_0^2 X = 0$ with $\omega_0 = \sqrt{\frac{k}{m}}$, which has the solution $X = x_0 \cdot \frac{p}{p^2+\omega_0^2}$. From $L4$ of the Laplace transform table (Boas p.469-471), you reach $x(t) = x_0 \cos\omega_0 t$.

• (b) $p^2X - px_0 + 2\beta pX - 2\beta x_0 + \omega_0^2 X = 0$ with $\beta = \frac{b}{2m}$. Therefore,

$$X = x_0 \cdot \frac{p+2\beta}{p^2+2\beta p+\omega_0^2} = x_0 \cdot \frac{p+2\beta}{(p+\beta)^2+\omega_1^2} = x_0 \left[\frac{p+\beta}{(p+\beta)^2+\omega_1^2} + \frac{\beta}{\omega_1}\frac{\omega_1}{(p+\beta)^2+\omega_1^2}\right]$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. From $L13$-$L14$ of the Laplace transform table, you can show that $x(t) = x_0 e^{-\beta t}[\cos\omega_1 t + \frac{\beta}{\omega_1}\sin\omega_1 t] = x_0 e^{-\beta t}\left(\frac{\omega_0}{\omega_1}\right)\cos(\omega_1 t - \delta)$ with $\tan\delta = \frac{\beta}{\omega_1}$.

• (c-1) Let us write the position of m as $\tilde{x}(t) = d(t) + x(t) = \frac{1}{2}at^2 + x(t)$. Inside the accelerating system (gray box in the figure), m experiences the fictitious force $-m\mathbf{a}$. Therefore, the equation of motion becomes $m\ddot{x}(t) + kx(t) = -ma$ with initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$ (note that $x(t)$ is *not* from its new equilibrium point; the spring was initially unstretched). The Laplace transform gives $p^2X + \omega_0^2 X = -\frac{a}{p} \;\rightarrow\; X = \frac{-a/p}{(p^2+\omega_0^2)} = -\frac{a}{\omega_0^2}\cdot\frac{\omega_0^2}{p(p^2+\omega_0^2)}$. From $L15$ of the Laplace transform table, you find $x(t) = -\frac{a}{\omega_0^2}(1-\cos\omega_0 t)$, thus $\tilde{x}(t) = \frac{1}{2}at^2 - \frac{a}{\omega_0^2}(1-\cos\omega_0 t)$.

• (c-2) $\tilde{x}(t) \simeq \frac{1}{2}at^2 - \frac{a}{\omega_0^2}(1 - 1 + \frac{\omega_0^2 t^2}{2!} - \frac{\omega_0^4 t^4}{4!}) = \frac{a\omega_0^2}{24}\cdot t^4$ for $\omega_0 t \ll 1$.

10. In the beginning of Boas Chapter 8, Section 7, the author discusses many methods of solving various types of second-order ODEs. Among them is Lagrange's *method of variation of parameters* to find a particular solution of an inhomogeneous ODE.

(a) Let us start with a homogeneous second-order linear ODE in the form of

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous functions of x. Let us assume that we know its two independent solutions, y_1 and y_2. Now, for the inhomogeneous second-order linear ODE of

$$y'' + p(x)y' + q(x)y = f(x),$$

show that a particular solution $y_p(x)$ is written as

$$y_p(x) = -y_1(x) \int \frac{y_2(x')f(x')}{W(x')} dx' + y_2(x) \int \frac{y_1(x')f(x')}{W(x')} dx'$$

where $W(x')$ is the Wronskian of y_1 and y_2, $W(y_1(x'), y_2(x'))$.

(Note: You may start with $y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$ and follow the step-by-step instruction given in Boas Chapter 8, Problem 12.14(b) that leads to the set of two conditions for c_1 and c_2: $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ and $c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = f(x)$. Notice that the first equation of this set is our "imposed" condition, while the second one is what you get if you plug $y_p' = \{c_1'(x)y_1(x) + c_2'(x)y_2(x)\} + \{c_1(x)y_1'(x) + c_2(x)y_2'(x)\} = c_1(x)y_1'(x) + c_2(x)y_2'(x)$ and the corresponding y_p'' into our ODE above. In case you wonder, no knowledge about the Green function in Section 12 is needed to tackle this problem.)

Now, utilizing the given solution of the homogeneous equation, find a solution of each of the following inhomogeneous ODEs. (More exercise problems in Chapter 8, Problems 12.15-18.)

(b) $y'' + y = \sec x$;   with $y_1 = \cos x$ and $y_2 = \sin x$

(c) $(1 - x)y'' + xy' - y = (1 - x)^2$;   with $y_1 = x$ and $y_2 = e^x$

- (b) $y(x) = C_1 \cos x + C_2 \sin x + y_p(x) = (C_1 + \ln|\cos x|)\cos x + (C_2 + x)\sin x$
- (c) $y(x) = C_1 x + C_2 e^x + y_p(x) = C_1' x + C_2 e^x + x^2 + 1$