Mathematical Physics I (Fall 2021): Homework #3 Solution

Due Oct. 15, 2021 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

1. Boas Chapter 4, Problem 7.28

(Note: For Problem 7.28, you may want to prove and utilize the findings in Problem 7.27. Note the meaning of the subscripts next to the partial derivatives from Boas Chapter 4, Section 1 in case you did not read it. Prove further that the resulting formula is the familiar $c_p - c_v = nR$ found in e.g., Schroeder or Halliday & Resnick, where n is the number of moles of gas present and R is the gas constant.)

• Replacing $x \to T$, $y \to v$, $z \to p$ and $u \to s$ gives $\left(\frac{\partial s}{\partial T}\right)_p = \left(\frac{\partial s}{\partial T}\right)_v + \left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_p$, thus $c_p = c_v + T\left(\frac{\partial s}{\partial v}\right)_T \left(\frac{\partial v}{\partial T}\right)_p$. With $\left(\frac{T\partial s}{\partial v}\right)_T = p$ and $\left(\frac{\partial v}{\partial T}\right)_p = \left(\frac{\partial(nRT/p)}{\partial T}\right)_p = \frac{nR}{p}$, you find $c_p - c_v = nR$.

2. Boas Chapter 4, Problem 8.8

(Note: For Problem 8.8, tackle the stated problem in two ways, one with the method in Section 8 and the other with the Lagrange multiplier described in Section 9.)

• The cross section of the gutter is $A = \frac{1}{2}b\sin\theta\{(l-2b) + (l - 2b + 2b\cos\theta)\} = b^2\sin\theta\cos\theta + (l-2b)b\sin\theta$, where $l = 24$ cm, b is the amount at each side that is bent up, and θ is the angle through which it is bent. Differentiating A with two independent variables θ and b, $\frac{\partial A}{\partial \theta} = 0$ gives $l - 2b + b\cos\theta - \frac{b\sin^2\theta}{\cos\theta} = 0$ while $\frac{\partial A}{\partial b} = 0$ gives $l - 4b + 2b\cos\theta = 0$. Combining the two equations, you reach $b = \frac{l}{3}$ and $\theta = \frac{\pi}{3}$.

• Alternatively, to utilize the Lagrange multiplier, we write $F = A + \lambda l$ from Eq.(9.6) with $A = b\sin\theta(a + b\cos\theta)$ and $l = a + 2b$. Differentiating F with three variables θ, a and b, $\frac{\partial F}{\partial \theta} = 0$ gives $a + b\cos\theta - \frac{b\sin^2\theta}{\cos\theta} = 0$, $\frac{\partial F}{\partial a} = 0$ gives $b\sin\theta + \lambda = 0$, and $\frac{\partial F}{\partial b} = 0$ gives $a + 2b\cos\theta - 2b = 0$. You can easily observe that the three equations here are equivalent to the two equations acquired by the first method.

3. Boas Chapter 4, Problem 10.10

• (b) From Eq.(9.6), we write $F = y^2 + xz + \lambda(x^2 + y^2 + z^2)$. Then, $\frac{\partial F}{\partial x} = z + 2\lambda x = 0$, $\frac{\partial F}{\partial z} = x + 2\lambda z = 0$, and $\frac{\partial F}{\partial y} = 2y + 2\lambda y = 0$. The last equation offers two possibilities of extremum conditions: (i) $\lambda = -1 \;\to\; x = z = 0 \;\to\; y = \pm 1$ (giving $T_{\text{max, local}} = 1$). (ii) $y = 0$

$\rightarrow\ x = \pm\frac{1}{\sqrt{2}} = z$ with $\lambda = -\frac{1}{2}$ (giving $T_{\text{max, local}} = \frac{1}{2}$), or $x = \pm\frac{1}{\sqrt{2}} = -z$ with $\lambda = \frac{1}{2}$ (giving $T_{\text{min, local}} = -\frac{1}{2}$). Note that case (ii) is essentially the problem (a).

4. Boas Chapter 5, Problem 5.5

• First, the intersection between the sphere and the cone needs to be found. Then from Eq.(5.2) with $\phi(x,y,z) = \phi(r,\theta,z) = 3r^2 - z^2 = 0$,

$$A = 2\iint \sec\gamma\, dxdy = 2\int_0^2 dr \int_0^{2\pi} rd\theta \cdot \frac{|\nabla\phi|}{|\partial\phi/\partial z|} = 4\pi\int_0^2 rdr \cdot \frac{\sqrt{36r^2+4z^2}}{|-2z|}$$
$$= 4\pi\int_0^2 rdr \cdot \frac{\sqrt{12z^2+4z^2}}{|-2z|} = 8\pi\int_0^2 rdr = 16\pi.$$

5. Boas Chapter 5, Problem 6.27

(Note: For Problem 6.27, you will first have to show that the intervals of integration for u and v are $[0,1]$ and $[0, 1+u]$, respectively. See the hint in Problem 4.20 for more information.)

• From Eq.(4.8) with $x = \frac{v}{1+u}$ and $y = \frac{uv}{1+u}$,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{-v}{(1+u)^2} & \frac{1}{1+u} \\ \frac{v}{(1+u)^2} & \frac{u}{1+u} \end{vmatrix} = -\frac{v}{(1+u)^2},$$

which makes the integral

$$\int_0^1 dx \int_0^x \frac{(x+y)e^{x+y}}{x^2}dy = \int_0^1 du \int_0^{1+u} \frac{ve^v}{(v/1+u)^2}|J|dv = \int_0^1 du \int_0^{1+u} e^v dv$$
$$= \left[e^{1+u} - u\right]\Big|_0^1 = e^2 - e - 1.$$

6. Boas Chapter 6, Problem 3.12

• (a) Using Thm.(3.9), $(\mathbf{A}\cdot\mathbf{B})^2 - [(\mathbf{A}\times\mathbf{B})\times\mathbf{B}]\cdot\mathbf{A} = (\mathbf{A}\cdot\mathbf{B})^2 - [-(\mathbf{B}\cdot\mathbf{B})\mathbf{A} + (\mathbf{A}\cdot\mathbf{B})\mathbf{B}]\cdot\mathbf{A} = (\mathbf{A}\cdot\mathbf{B})^2 + A^2B^2 - (\mathbf{A}\cdot\mathbf{B})^2 = A^2B^2$.

• (b-1) Using Eq.(3.3) and Thm.(3.9), $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = \mathbf{A}\cdot[\mathbf{B}\times(\mathbf{C}\times\mathbf{D})] = \mathbf{A}\cdot[(\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{B}\cdot\mathbf{C})\mathbf{D}] = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$. You may prove the identity in a brute force approach using Eq.(3.2), too.

• (b-2) Alternatively, writing in tensor notation and then using Eqs. (5.8) and (5.11) of Boas Chapter 10, $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\times\mathbf{B})_i(\mathbf{C}\times\mathbf{D})_i = \epsilon_{ijk}A_jB_k \cdot \epsilon_{imn}C_mD_n = (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})A_jB_kC_lD_m = A_jB_kC_jD_k - A_jB_kC_kD_j = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$.

7. Boas Chapter 6, Problem 6.16

• Using Eq.(9.6) of Chapter 4 and Eq.(6.2) of Chapter 6, we write $F = \frac{d\phi}{ds} + \lambda(a^2+b^2+c^2) = \left(\frac{\partial\phi}{\partial x}a + \frac{\partial\phi}{\partial y}b + \frac{\partial\phi}{\partial z}c\right) + \lambda(a^2+b^2+c^2)$. Then, $\frac{\partial F}{\partial a} = \frac{\partial\phi}{\partial x} - 2\lambda a = 0$, $\frac{\partial F}{\partial b} = \frac{\partial\phi}{\partial y} - 2\lambda b = 0$,

$\frac{\partial F}{\partial c} = \frac{\partial \phi}{\partial z} - 2\lambda c = 0$, and $a^2 + b^2 + c^2 = 1$. Combined, $\lambda = \pm\frac{|\nabla\phi|}{2}$ and $\mathbf{u} = (a, b, c) = \frac{\nabla\phi}{2\lambda} = \pm\frac{\nabla\phi}{|\nabla\phi|}$. With Eq.(6.4), you get the extremum values of $\frac{d\phi}{ds}$ as $\nabla\phi \cdot \mathbf{u} = \pm\frac{(\nabla\phi)^2}{|\nabla\phi|} = \pm|\nabla\phi|$.

8. Boas Chapter 6, Problem 9.8

(Note: For Problem 9.8, you will need to prove and utilize the findings in Problem 9.6.)

• In an approach similar to Problem 9.7, the area inside the curve can be written with $x = 8\cos^3\theta$ and $y = 8\sin^3\theta$ as

$$\begin{aligned}\iint dxdy &= \frac{1}{2}\oint(8\cos^3\theta \cdot 24\sin^2\theta\cos\theta + 8\sin^3\theta \cdot 24\cos^2\theta\sin\theta)d\theta \\ &= 96\int_0^{2\pi}\sin^2\theta\cos^2\theta\, d\theta = 24\int_0^{2\pi}\sin^2 2\theta\, d\theta = 24\pi.\end{aligned}$$

9. The ground state energy of a quantum particle of mass m in a *right circular cylinder* of radius R and height H is given by

$$E = \frac{\hbar^2}{2m}\left(\frac{k_{10}^2}{R^2} + \frac{\pi^2}{H^2}\right), \tag{1}$$

where k_{mn} is the mth positive zero of the Bessel function $J_n(x)$ (k_{mn} defined right before Eq.(5.17) in Boas Chapter 13; their numerical values such as $k_{10} = 2.4048$ can be found in e.g., http://mathworld.wolfram.com/BesselFunctionZeros.html). After briefly reviewing what each term of Eq.(1) means, find the ratio of R to H that minimizes the energy for a fixed cylinder volume.

(Note: The meaning of the second term in Eq.(1) should be obvious from the *particle in a box* problem in rudimentary quantum mechanics (i.e., quantum particle in an infinite potential well; see e.g., Halliday & Resnick). For the first term, you may want to briefly look through the textbooks in quantum mechanics, or the description of its classical analog in Boas Chapter 13.6. Once again, you are simply asked to come up with 2-3 sentences about what Eq.(1) means, without laboriously deriving the equation in great detail. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

• In Eq.(9.6) of Boas Chapter 4, we set $f(R, H) = \frac{2mE}{\hbar^2} = \frac{k_{10}^2}{R^2} + \frac{\pi^2}{H^2}$ and $\phi(R, H) = \pi R^2 H$. Differentiating $F(R, H) = f(R, H) + \lambda\phi(R, H)$ with respect to R and H gives $-\frac{2k_{10}^2}{R^3} + 2\lambda\pi RH = 0$ and $-\frac{2\pi^2}{H^3} + \lambda\pi R^2 = 0$. Combining the two equations one gets $-\lambda\pi R^2 H = \frac{k_{10}^2}{R^2} = \frac{2\pi^2}{H^2} \rightarrow$ therefore, $\frac{R}{H} = \frac{k_{10}}{\pi\sqrt{2}} = \frac{2.4048}{\pi\sqrt{2}} = 0.5413$.

10. (a) Review Boas Chapter 6, Example 7.2, thus prove Eq.(7.6) or Eq.(f) in p.339.

(b) Review how Gauss's law $\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$ is acquired from Coulomb's law in Boas Chapter 6, Section 10, where $\mathbf{E}$ is the electric field, ρ is the volume charge density, and ϵ_0 is the permittivity of free space.

(c) Prove that the energy of a continuous charge distribution is given by

$$\frac{1}{2}\int \rho V d\tau = \frac{\epsilon_0}{2}\int |\mathbf{E}|^2 d\tau$$

for integration over all space, where V is the electrostatic potential. You are first asked to briefly discuss how one came up with the term $\frac{1}{2}\int \rho V d\tau$. To prove the equality, you may assume that V vanishes at large distance r at least as fast as r^{-1}.

(Note: The exercises here should sound familiar to most of you as you have begun to study and explore electromagnetism. If not, you may want to briefly review the classic textbooks in electromagnetism such as Griffiths or Jackson.)

- (a) Another way to prove Eq.(7.6) can be found in Section 1.2.6 of Griffiths (4th ed.).
- (c) One can think of $\frac{1}{2}\int \rho V d\tau$ as a continuum approximation of $\frac{1}{2}\sum_i q_i V(\mathbf{r}_i)$. See Section 2.4.2 of Griffiths. Using (a) and (b), one gets

$$\begin{aligned} W &= \frac{1}{2}\int \rho V d\tau = \frac{\epsilon_0}{2}\int V(\nabla\cdot\mathbf{E})d\tau = \frac{\epsilon_0}{2}\int \left[-\mathbf{E}\cdot(\nabla V) + \nabla\cdot(V\mathbf{E})\right] d\tau \\ &= \frac{\epsilon_0}{2}\left(\int |\mathbf{E}|^2 d\tau + \oint_{\partial\tau} (V\mathbf{E})\cdot\mathbf{n}d\sigma\right). \end{aligned}$$

The second integral vanishes as the integration is carried out over all space — since it behaves like $(r^{-1}r^{-2})\cdot r^2 = r^{-1}$ which tends to 0 as r approaches ∞.