Mathematical Physics I (Fall 2021): Homework #1 Solution

Due Sep. 17, 2021 (Fri, 23:00pm)

[0.5 pt each, total 5 pts]

1. Boas Chapter 1, Problem 6.6

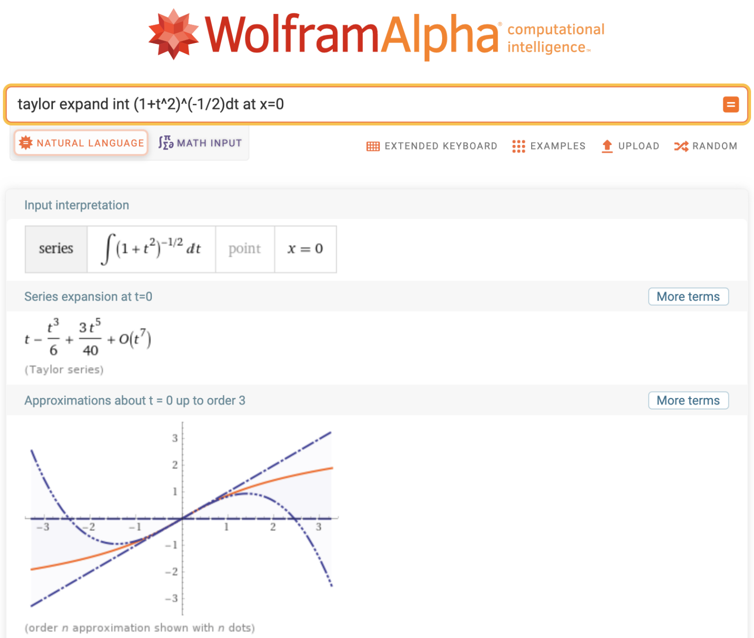
(Note: For Problem 6.6, compare with Oresme's proof demonstrated in Problem 6.2.)

• The series diverges according to the comparison test.

2. Boas Chapter 1, Problem 13.19

(Note: For Problem 13.19, first prove the equality given, for which you may need to prove and utilize $\sec\theta = \frac{d(\sec\theta+\tan\theta)/d\theta}{\sec\theta+\tan\theta}$.)

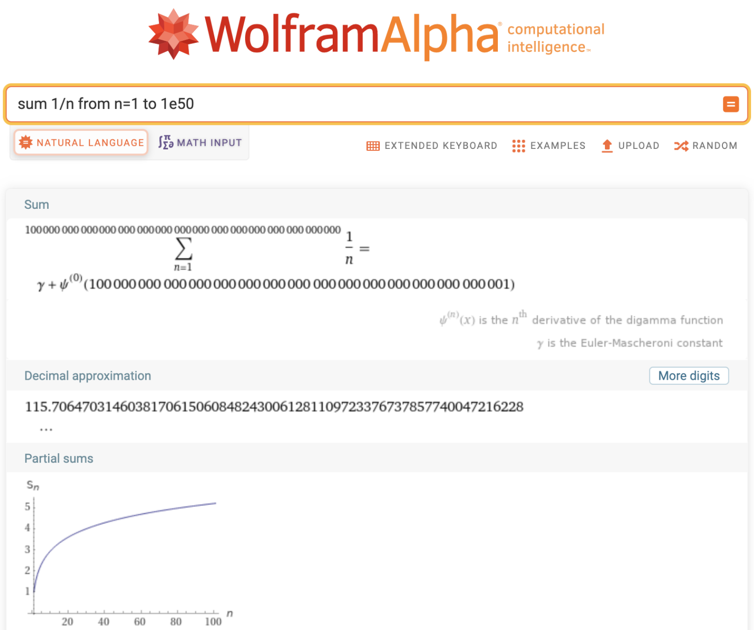
• From Eq.(13.5), you can get the binomial series expansion $(1+t^2)^{-\frac{1}{2}} \simeq 1+(-\frac{1}{2})t^2+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}t^4$. Integrating both sides, you find $\int_0^x (1+t^2)^{-\frac{1}{2}} dt \simeq x-\frac{1}{6}x^3+\frac{3}{40}x^5$. In a manner similar to Example 13.C2 or Problem 13.4, the general terms are found as $\int_0^x (1+t^2)^{-\frac{1}{2}} dt = \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{(2n)!!(2n+1)} x^{2n+1}$.



3. Boas Chapter 1, Problem 16.1

(Note: For Problem 16.1, it is helpful to draw a diagram that considers the balance between forces and torques, a so-called *extended free-body diagram.*)

• You may find it interesting to read an article about the stacking problem such as Hall, J., 2005, "Fun with stacking block", American Journal of Physics, 73, 1107.



4. Boas Chapter 2, Problem 7.13

• $\rho = \lim_{n\to\infty} \left|\frac{n(z-i)}{n+1}\right| = |z-i| < 1.$

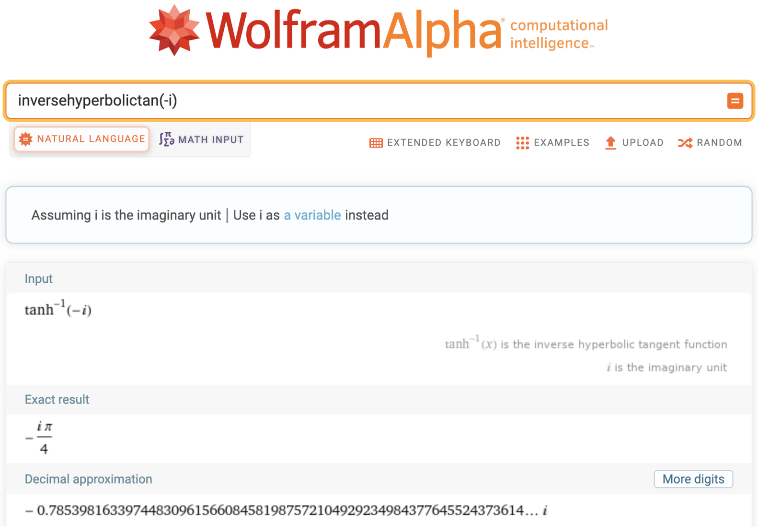
5. Boas Chapter 2, Problem 12.27

• Using Eqs.(11.4) and (12.2), $\sin(4+3i) = -\frac{i}{2}(e^{-3+4i} - e^{3-4i}) = -\frac{i}{2}(e^{-3}\cos 4 - e^{3}\cos 4) + \frac{1}{2}(e^{-3}\sin 4 - e^{3}\sin 4) = i\cos 4 \sinh 3 + \sin 4 \cosh 3.$

6. Boas Chapter 2, Problem 15.6

(Note: For Problem 15.6, read the instruction in the textbook carefully; that is, you have been asked to compare your result with a computer solution as well. Resolve or explain any disagreement in results.)

• With $u \equiv e^z$, you get $\tanh z = -i = \frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{u - u^{-1}}{u + u^{-1}} \rightarrow u^2 = \frac{1-i}{1+i} = -i \rightarrow z = \ln u = \ln[\pm(-i)^{\frac{1}{2}}] = \ln[e^{i(\frac{3}{2}\pi + 2n\pi)\cdot\frac{1}{2}}] = i(n\pi + \frac{3}{4}\pi).$



7. Boas Chapter 2, Problem 17.19

• $\arctan z = \omega \;\rightarrow\;$ With $u \equiv e^{i\omega}$, you find $z = \tan\omega = \frac{e^{i\omega}-e^{i\omega}}{i(e^{i\omega}+e^{i\omega})} = \frac{u-u^{-1}}{i(u+u^{-1})} \;\rightarrow\; u^2 = \frac{1+iz}{1-iz} \;\rightarrow\; \omega = \frac{1}{i}\ln u = \frac{1}{i}\ln\left(\frac{1+iz}{1-iz}\right)^{\frac{1}{2}} = \arctan z$.

8. Boas Chapter 3, Problem 2.14

(Note: For Problem 2.14, again, read the instruction in the textbook carefully.)

• The equations are inconsistent, so no solution exists.

9. In the quantum statistical distribution of bosons with unspecified total number of particles (so-called Bose-Einstein statistics of quantized oscillators), the average energy of the system is found to be

$$\bar{\epsilon} = \frac{\sum_{n=1}^{\infty} n\epsilon_0 e^{-\frac{n\epsilon_0}{k_B T}}}{\sum_{n=0}^{\infty} e^{-\frac{n\epsilon_0}{k_B T}}}, \tag{1}$$

where ϵ_0 is a fixed energy, k_B is the Boltzmann constant, and T is the temperature. (a) After briefly discussing how this formula is acquired, show that the ratio becomes

$$\bar{\epsilon} = \frac{\epsilon_0}{e^{\frac{\epsilon_0}{k_\mathrm{B}T}} - 1}$$

by first identifying Eq.(1)'s denominator as a binomial expansion of $\frac{1}{1-x}$ with $x = e^{-\frac{\epsilon_0}{k_\mathrm{B}T}}$, and its numerator as a constant times $\frac{x}{(1-x)^2}$. (b) Using a power series expansion, show that $\bar{\epsilon}$ reduces to $k_\mathrm{B}T$ in the classical limit of $k_\mathrm{B}T \gg \epsilon_0$.

(Note: You may want to briefly review the classic textbooks in statistical mechanics such as Schroeder or Reif. You are simply asked to come up with a short paragraph about what Eq.(1) means, without laboriously deriving the equation in great detail. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

- (a) By differentiating $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, you find $\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$, which leads to

$$\bar{\epsilon} = \frac{\sum_{n=1}^{\infty} \epsilon_0 n x^n}{\sum_{n=0}^{\infty} x^n} = \frac{\epsilon_0 x}{1-x} = \frac{\epsilon_0}{x^{-1}-1}.$$

- (b) With $y = \frac{\epsilon_0}{k_\mathrm{B}T} \ll 1$, you get $\bar{\epsilon} = \frac{\epsilon_0}{e^y - 1} \simeq \frac{\epsilon_0}{1+y-1} = k_\mathrm{B}T$.

10. By using a complex power series, prove that

$$\sum_{n=0}^{N-1} \cos nx = \frac{\sin\frac{Nx}{2}\,\cos\frac{(N-1)x}{2}}{\sin\frac{x}{2}} \quad \text{and} \quad \sum_{n=0}^{N-1} \sin nx = \frac{\sin\frac{Nx}{2}\,\sin\frac{(N-1)x}{2}}{\sin\frac{x}{2}}.$$

Discuss briefly how these series appear in the description of multiple-slit diffraction patterns.

(Note: You may want to review the freshman physics textbooks such as Halliday & Resnick.)

- $$\sum_{n=0}^{N-1} (e^{ix})^n = \frac{1-e^{iNx}}{1-e^{ix}} = \frac{e^{iNx}-1}{e^{ix}-1} = \frac{e^{iNx/2}}{e^{ix/2}} \cdot \frac{e^{iNx/2} - e^{-iNx/2}}{e^{ix/2} - e^{-ix/2}} = e^{i(N-1)x/2} \cdot \frac{\sin\frac{Nx}{2}}{\sin\frac{x}{2}},$$

yielding results that are very similar to one of the Examples in Boas Chapter 2, Section 16.