Mathematical Physics I (Fall 2021): Homework #6

Due Dec. 3, 2021 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as Matlab or Mathematica.

1.-2. Boas Chapter 8, Problems 11.8, 12.6

(Note: For Problem 11.8, you will first want to review Problem 11.6(b) and (c), and Example 10.1. You may assume $y_0 = y_0' = 0$ as in Problem 11.6(a).)

3.-5. Boas Chapter 13, Problems 2.7, 3.3, 4.5

(Note: For Problem 2.7, you will first have to work out Problem 2.3. For Problem 3.3, you may want to review Chapter 13, Example 3.2.)

6.-8. Boas Chapter 9, Problems 2.10, 3.7, 4.4

(Note: For Problem 3.7, you *must* change the independent variable; then, compare your answer with that of Problem 2.5. You may want to utilize the indefinite integral found in Problem 13.19 of Boas Chapter 1 (— along with the formula in Problem 17.20 of Chapter 2), or in extensive references like Zwillinger (Sections 5.4.1 and 6.11.1, 33rd ed.). For Problem 4.4, as you evaluate the integral for T you may simply utilize the indefinite integral $\int \frac{dx}{\sqrt{ax^2+bx+c}} = -\frac{1}{\sqrt{-a}}\sin^{-1}(\frac{2ax+b}{\sqrt{b^2-4ac}})$ (if $a < 0$ & $b^2 > 4ac$) found in common integral tables like Appendix E of Thornton & Marion, or in extensive references like Zwillinger (Section 5.4.13, 33rd ed.).)

9. In the class we briefly discussed that the crucial properties of the Dirac delta function $\delta(x)$ may be developed as the limiting case using any of the following sequences of functions:

$$\delta_{n,1}(x) = \begin{cases} n, & \text{if } |x| < \frac{1}{2n} \\ 0, & \text{otherwise,} \end{cases} \qquad \delta_{n,2}(x) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2},$$

$$\delta_{n,3}(x) = \frac{n}{\pi} \frac{1}{1+n^2x^2}, \qquad \delta_{n,4}(x) = \frac{n}{2\cosh^2 nx},$$

$$\delta_{n,5}(x) = \frac{\sin nx}{\pi x}, \qquad \delta_{n,6}(x) = \frac{1}{2\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin \frac{1}{2}x}, \quad \text{etc.}$$

In other words, we may regard $\delta(x)$ as a normalized *distribution* which is defined with

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx. \tag{1}$$

(a) Using $\delta_{n,1}(x)$ to $\delta_{n,5}(x)$, show that

$$\int_{-\infty}^{\infty} \delta_n(x) dx = 1 \quad \text{for all } n.$$

(b) Using $\delta_{n,1}(x)$ and $\delta_{n,5}(x)$, prove

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0).$$

(c) Treating $\delta_n(x)$ and its derivative as in Eq.(1), prove one of the properties of $\delta(x)$,

$$x\delta'(x) = -\delta(x).$$

(Note: For $\delta_{n,3}(x)$ in (a), you may need to prove and use the indefinite integral $\int \frac{dx}{1+x^2} = \arctan x$. For $\delta_{n,4}(x)$, you may simply utilize the indefinite integral $\int \text{sech}^2 x\, dx = \tanh x$ found in common integral tables, or in extensive references like Zwillinger (Section 5.4.20, 33rd ed.). For $\delta_{n,5}(x)$, you may also take advantage of the definite integral $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$ found in Boas Chapter 7, Example 12.2 or in Chapter 14, Example 7.4.)

10. Let us apply the variational calculus to a few physics examples.

(a) Show that the Lagrangian of a particle of rest mass m in a motion along x,

$$L = mc^2(1 - \sqrt{1-\beta^2}) - V(x),$$

with $\beta = \frac{v}{c}$ leads to a relativistic form of Newton's second law of motion,

$$F = m\frac{dv}{dt}(1-\beta^2)^{-\frac{3}{2}}$$

where t and $v = \frac{dx}{dt}$ are the proper time and velocity, respectively.

(b) Show that the Lagrangian of a particle of mass m and charge q in an electromagnetic field described by scalar potential V and vector potential $\mathbf{A}$,

$$L = \frac{1}{2}mv^2 - qV + q\mathbf{A}\cdot\mathbf{v},$$

leads to a form of Newton's second law of motion for the charged particle,

$$m\frac{dv_i}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})_i$$

with $i = 1, 2, 3$ representing a 3-dimensional motion.

(Note: The equations here should look familiar to most of you as you have covered classical mechanics, special relativisty, and electromagnetism. If not, you may want to briefly review the textbooks in classical mechanics such as Thornton & Marion or Taylor, and in electromagnetism such as Griffiths or Jackson. For (b), you may need to use the basic relations in electromagnetism, $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$, seen in Section 10.1.1 of Griffiths (4th ed.). You will also have to utilize $\frac{dA_i}{dt} = \frac{\partial A_i}{\partial t} + \sum_{j=1}^{3} \dot{x}_j \frac{\partial A_i}{\partial x_j}$ which is found in Boas Chapter 6, Problems 10.15 and 12.3, or can be easily proven from the chain rule discussed in Chapter 4, Section 5.)