Mathematical Physics I (Fall 2021): Homework #5

Due Nov. 19, 2021 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.

1.-8. Boas Chapter 8, Problems 5.9, 6.8, 6.14, 6.35, 7.3, 7.17, 8.11, 9.14

(Note: For Problems in Sections 5 and 6, read the instruction in the textbook carefully; that is, you have been asked to find a computer solution and reconcile differences, if any. You may continue to utilize computer solutions to validate your answers to problems in Sections 7 to 9.)

9. Let us apply the Laplace transform to a few physical examples.

(a) Consider a mass m attached to one end of an ideal, massless spring of spring constant k. The free end of the spring is fixed in space and the mass is oscillating under the spring's influence. The displacement $x(t)$ from the equilibrium point (see figure) satisfies the equation of motion

$$m\ddot{x}(t) + kx(t) = 0,$$

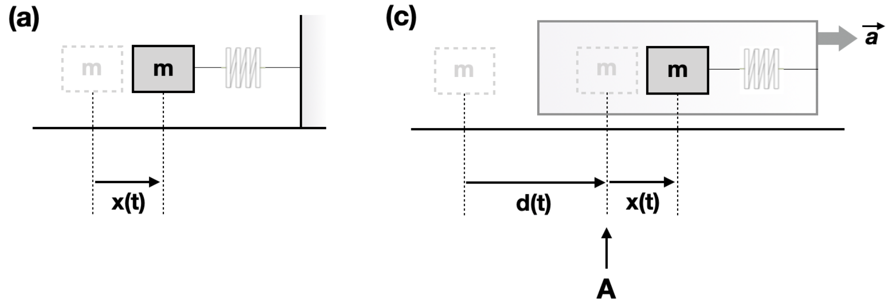
with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Using the Laplace transforms, find $x(t)$.

(b) We then subject the system in (a) to the damping proportional to the velocity as

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0,$$

with initial conditions $x(0) = x_0$ and $\dot{x}(0) = 0$. Using the Laplace transforms, recover $x(t)$ that you found in classical mechanics classes, e.g., Eq.(3.40) of Thornton & Marion (5th ed.). Assume underdamped oscillation, i.e., $b^2 < 4mk$.

(c) We now go back to the setup in (a) again. Starting at $t = 0$, the free end of the unstretched spring experiences a constant acceleration **a**, away from the mass m at rest. Using the Laplace transforms, find the position of m. Also find its form in the limit of small t.



(Note: The differential equations here should look familiar to most of you from your classical mechanics classes. If not, you may want to briefly review the textbooks such as Thornton & Marion or Taylor. For (c), the motion of m can be divided into two parts: the accelerated motion of point A defined as where m would have been had the spring been replaced by a string (see figure) + the oscillation of m about A — expressed as $d(t)$ and $x(t)$, respectively.)

10. In the beginning of Boas Chapter 8, Section 7, the author discusses many methods of solving various types of second-order ODEs. Among them is Lagrange's *method of variation of parameters* to find a particular solution of an inhomogeneous ODE.

(a) Let us start with a homogeneous second-order linear ODE in the form of

$$y'' + p(x)y' + q(x)y = 0$$

where p and q are continuous functions of x. Let us assume that we know its two independent solutions, y_1 and y_2. Now, for the inhomogeneous second-order linear ODE of

$$y'' + p(x)y' + q(x)y = f(x),$$

show that a particular solution $y_p(x)$ is written as

$$y_p(x) = -y_1(x) \int \frac{y_2(x')f(x')}{W(x')} dx' + y_2(x) \int \frac{y_1(x')f(x')}{W(x')} dx'$$

where $W(x')$ is the Wronskian of y_1 and y_2, $W(y_1(x'), y_2(x'))$.

(Note: You may start with $y_p = c_1(x)y_1(x) + c_2(x)y_2(x)$ and follow the step-by-step instruction given in Boas Chapter 8, Problem 12.14(b) that leads to the set of two conditions for c_1 and c_2: $c_1'(x)y_1(x) + c_2'(x)y_2(x) = 0$ and $c_1'(x)y_1'(x) + c_2'(x)y_2'(x) = f(x)$. Notice that the first equation of this set is our "imposed" condition, while the second one is what you get if you plug $y_p' = \{c_1'(x)y_1(x) + c_2'(x)y_2(x)\} + \{c_1(x)y_1'(x) + c_2(x)y_2'(x)\} = c_1(x)y_1'(x) + c_2(x)y_2'(x)$ and the corresponding y_p'' into our ODE above. In case you wonder, no knowledge about the Green function in Section 12 is needed to tackle this problem.)

Now, utilizing the given solution of the homogeneous equation, find a solution of each of the following inhomogeneous ODEs. (More exercise problems in Chapter 8, Problems 12.15-18.)

(b) $y'' + y = \sec x$; with $y_1 = \cos x$ and $y_2 = \sin x$

(c) $(1 - x)y'' + xy' - y = (1 - x)^2$; with $y_1 = x$ and $y_2 = e^x$