Mathematical Physics I (Fall 2021): Homework #4

Due Nov. 5, 2021 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.

1.-6. Boas Chapter 7, Problems 5.3, 6.12 (for Problem 5.7), 7.7, 8.15(b), 9.10, 12.23

(Note: For Problems 6.12 and 7.7, you will first have to work out Problem 5.7. Then for Problem 6.12 you may consider a way to automatically draw several partial sums of different n's — e.g., a simple script written in MATLAB or in `python`. For Problem 12.23, you are asked to first work out Problem 12.17.)

7.-8. Boas Chapter 8, Problems 2.8, 3.4

(Note: For Problem 2.8, read the instruction in the textbook carefully; that is, you have been asked to plot a slope field with a computer, for example. Read Boas Chapter 8, Section 1 to learn about the "slope field". For Problem 3.4, again, read the instruction in the textbook carefully.)

9. In one of the methods to convert AC voltage into DC voltage, a *full wave rectifier* yields the waveform $|\sin \omega_0 t|$ — i.e., passes the positive peaks of the incoming sine wave but inverts the negative peaks.

(a) After briefly discussing how a full wave rectifier works, expand the rectified sine wave in an appropriate Fourier series. Show that all the odd harmonics are absent (as expected), and the lowest oscillation frequency is $2\omega_0$.

(b) Using the Fourier series found above, prove that

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\cdots=\sum_{k=1}^{\infty}\frac{1}{4k^2-1}=\frac{1}{2} \quad \text{and}$$

$$\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\frac{1}{7\cdot 9}+\cdots=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{4k^2-1}=\frac{\pi-2}{4}.$$

(Note: To learn more about how a rectifier works in an electronic circuit, see e.g., *The Art of Electronics* by Horowitz & Hill.)

10. Let us use the Fourier transform to appreciate the meaning of the Heisenberg uncertainty principle in quantum mechanics.

(a) Imagine an infinite wave train $\sin\omega_0 t$ clipped by shutters to maintain only N cycles of the original waveform:

$$f(t)=\begin{cases}\sin\omega_0 t, & \text{if } |\omega_0 t|<N\pi,\\ 0, & \text{if } |\omega_0 t|>N\pi.\end{cases}$$

Find the amplitude function of the Fourier (exponential) transform, $g(\omega)$. Since the prefactor may depend on the exact definition of the transform, do not worry too much about it.

(b) Find the amplitude function of the Fourier sine transform, $g_s(\omega)$. Again, since the prefactor may depend on the exact definition of the transform, do not worry too much about it.

(c) Show that, in the special case of $N=\frac{1}{2}$ and $\omega_0=1$, $g_s(\omega)$ becomes

$$g_{s,1}(\omega)=\sqrt{\frac{2}{\pi}}\,\frac{\omega\cos\left(\frac{\omega\pi}{2}\right)}{1-\omega^2}.$$

(d) Now consider the limit of $\omega_0\gg 1$ and $\omega\approx\omega_0$. Show that $g_s(\omega)$ is approximately equal to

$$g_{s,2}(\omega)=\frac{1}{\sqrt{2\pi}}\frac{\sin[(\omega_0-\omega)\frac{N\pi}{\omega_0}]}{\omega_0-\omega}.$$

Sketch or computer plot this function for $N=1,3,5,10$ and an arbitrary ω_0. Notice that for $N\gg 1$, $g_{s,2}(\omega)$ may be interpreted as proportional to what we later define as the Dirac delta function, δ, in Boas Chapter 8, Section 11.

(Note: You do not need to provide a mathematically rigorous proof that $g_{s,2}(\omega)$ indeed becomes proportional to $\delta(\omega-\omega_0)$ as $N\to\infty$. For now, observe the shape of $g_{s,2}(\omega)$ as you vary N, and based on your observation simply argue that $\lim_{\omega\to\omega_0} g_{s,2}(\omega)\to\infty$ as $N\to\infty$.)

(e) Show that the first zeros of $g_{s,2}(\omega)$ from $\omega=\omega_0$ are at $\omega=\omega_0\pm\Delta\omega=\omega_0\pm\frac{\omega_0}{N}$. Justify that $\Delta\omega=\frac{\omega_0}{N}$ could be a good measure of the spread (or uncertainty) in frequency of our clipped wave train. Then, establish the inverse relationship between the wave train's pulse length ($N\pi$) and the frequency spread ($\Delta\omega$). Finally, using the relationship along with the assumed *wave nature of matter*, explain the uncertainty principle of quantum mechanics.