Mathematical Physics I (Fall 2021): Homework #2

Due Oct. 1, 2021 (Fri, 23:00pm)

[0.5 pt each, total 5 pts, turn in as a single pdf file to eTL]

• By turning in your homework, you acknowledge that you have not received any unpermitted aid, nor have compromised your academic integrity during its preparation. (Remember the SNU College of Natural Sciences Honor Code!)

• Textbook problem numbering convention (Boas, 3rd ed.): [Section].[Problem No.]. Many problems in this assignment are from the list of suggested problems announced in the class.

• Exhibit all intermediate steps to receive full credits. Only handwritten answers are accepted except for numerical problems or verifications — for which you print out and turn in not just the end results (e.g., plots) but also the source codes in your favorite tools such as MATLAB or MATHEMATICA.

1.-8. Boas Chapter 3, Problems 5.35, 6.21, 7.27, 8.15, 9.17, 11.42, 11.60, 12.16

(Note: For Problem 6.21, compare your inverse of the coefficient matrix with the one found by the *Gauss-Jordan elimination method* and the one found with a computer. For Problem 11.60, you will need to prove and utilize the findings in Problem 11.57, or Eq.(11.36).)

9. The following three 2×2 matrices are used in the quantum mechanical description of the particles of spin 1/2 in three dimensions:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(a) After briefly discussing how these so-called Pauli spin matrices are introduced in quantum mechanics, demonstrate first that these matrices are both Hermitian and unitary.

Then show that they satisfy the following relations:

(b) $(\sigma_j)^2 = I_2$ where I_2 is the 2×2 unit matrix,

(c) $\sigma_j \sigma_k = i\sigma_l$ if $j, k, l = 1, 2, 3$ or a cyclic permutation thereof,

(d) $\sigma_j \sigma_k = \delta_{jk} I_2 + i \sum_l \epsilon_{jkl} \sigma_l$ where δ_{jk} and ϵ_{jkl} are defined in Eq.(9.4) of Boas Chapter 3 and in Eq.(5.3) of Chapter 10, respectively,

(e) $\sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}I_2$,

(f) $[\sigma_j, \sigma_k] \equiv \sigma_j\sigma_k - \sigma_k\sigma_j = 2i\sum_l \epsilon_{jkl}\sigma_l$ (or $[\sigma_j, \sigma_k] = 2i\sigma_l$ if $j, k, l = 1, 2, 3$ or a cyclic permutation thereof), which is called the fundamental commutation relation for angular momentum matrices.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. You are simply asked to come up with 2-3 sentences about how the matrices are used, without diving into laborious quantum mechanical derivations. If needed, you must reference your sources appropriately with a proper citation convention, but your answer must still be your own work in your own words. To access the electronic resources — e.g., academic journals — off-campus via SNU library's proxy service, see http://library.snu.ac.kr/using/proxy.)

10. In the class we discussed the Gram-Schmidt orthonormalization process for a linear vector space and for a general vector space.

(a) In one example we briefly examined but left for your exercise (Example 14.6 of Boas Chapter 3), with an inner product defined as

$$\langle f|g\rangle = \int_{-1}^{1} f^*(x)g(x)dx,$$

one can construct a set of orthonormal polynomials P_i that satisfy the orthonormality condition on the interval $-1 \leq x \leq 1$,

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \delta_{mn}$$

(see also Eq.(8.4) of Chapter 12, but note a different normalization factor). We later identify this set of functions as *Legendre polynomials*. Starting from Eq.(14.10), follow the procedure step by step and find for yourself the first four members of P_i.

(b) In a similar manner, with an inner product defined as

$$\langle f|g\rangle = \int_{0}^{\infty} f^*(x)g(x)e^{-x}dx,$$

find the first three members in the set of orthonormal polynomials L_i that satisfy the orthonormality condition on the interval $0 \leq x \leq \infty$,

$$\int_{0}^{\infty} L_m(x)L_n(x)e^{-x}dx = \delta_{mn}$$

(see also Eq.(22.22) of Chapter 12). We call this set of functions as *Laguerre polynomials*.

(c) Discuss briefly where the (associated) Legendre polynomials and the (associated) Laguerre polynomials appear in quantum mechanics or in other physics research.

(Note: You may want to briefly review the textbooks in quantum mechanics such as Griffiths & Schroeter. Once again, you are simply asked to come up with 2-3 sentences about how these polynomials are used, not the detailed quantum mechanical discussions.)